Homework 9

Due Tuesday 12/1/2009 at 5:30pm in my box in physics. These may also be handed in at the end of Justin Mitchell's office hours in PHYS 228 from 4:00-5:30pm Tuesday.

Fowles Problems

7.1

7.3

7.8

7.9

7.12

7.14

7.16

7.22

7.26

We conclude that ion rockets are not useful for launching payloads from Earth but are suitable for deep space missions starting from Earth orbit in which efficient but gentle propulsion systems can be used.

Problems

7.1 A system consists of three particles, each of unit mass, with positions and velocities as follows:

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{i} + \mathbf{j} & \mathbf{v}_1 &= 2\mathbf{i} \\ \mathbf{r}_2 &= \mathbf{j} + \mathbf{k} & \mathbf{v}_2 &= \mathbf{j} \\ \mathbf{r}_3 &= \mathbf{k} & \mathbf{v}_3 &= \mathbf{i} + \mathbf{j} + \mathbf{k} \end{aligned}$$

Find the position and velocity of the center of mass. Find also the linear momentum of the system.

7.2 (**a**) Find the kinetic energy of the system in Problem 7.1.
(**b**) Find the value of $mv_{cm}^2/2$.
(**c**) Find the angular momentum about the origin.

7.3 A bullet of mass m is fired from a gun of mass M. If the gun can recoil freely and the muzzle velocity of the bullet (velocity relative to the gun as it leaves the barrel) is v_0, show that the actual velocity of the bullet relative to the ground is $v_0/(1 + \gamma)$ and the recoil velocity for the gun is $-\gamma v_0/(1 + \gamma)$, where $\gamma = m/M$

7.4 A block of wood rests on a smooth horizontal table. A gun is fired horizontally at the block and the bullet passes through the block, emerging with half its initial speed just before it entered the block. Show that the fraction of the initial kinetic energy of the bullet that is lost as frictional heat is $\frac{3}{4} - \frac{1}{4}\gamma$, where γ is the ratio of the mass of the bullet to the mass of the block ($\gamma < 1$).

7.5 An artillery shell is fired at an angle of elevation of 60° with initial speed v_0. At the uppermost part of its trajectory, the shell bursts into two equal fragments, one of which moves directly upward, relative to the ground, with initial speed $v_0/2$. What is the direction and speed of the other fragment immediately after the burst?

7.6 A ball is dropped from a height h onto a horizontal pavement. If the coefficient of restitution is ϵ, show that the total vertical distance the ball goes before the rebounds cease is $h(1 + \epsilon^2)/(1 - \epsilon^2)$. Find also the total length of time that the ball bounces.

7.7 A small car of a mass m and initial speed v_0 collides head-on on an icy road with a truck of mass $4m$ going toward the car with initial speed $\frac{1}{2}v_0$. If the coefficient of restitution in the collision is $\frac{1}{4}$, find the speed and direction of each vehicle just after colliding.

7.8 Show that the kinetic energy of a two-particle system is $\frac{1}{2}mv_{cm}^2 + \frac{1}{2}\mu v^2$, where $m = m_1 + m_2$, v is the relative speed, and μ is the reduced mass.

7.9 If two bodies undergo a direct collision, show that the loss in kinetic energy is equal to

$$\tfrac{1}{2}\mu v^2(1 - \epsilon^2)$$

where μ is the reduced mass, v is the relative speed before impact, and ϵ is the coefficient of restitution.

7.10 A moving particle of mass m_1 collides elastically with a target particle of mass m_2, which is initially at rest. If the collision is head-on, show that the incident particle loses a fraction $4\mu/m$ of its original kinetic energy, where μ is the reduced mass and $m = m_1 + m_2$.

7.11 Show that the angular momentum of a two-particle system is

$$\mathbf{r}_{cm} \times m\mathbf{v}_{cm} + \mathbf{R} \times \mu\mathbf{v}$$

where $m = m_1 + m_2$, μ is the reduced mass, $\mathbf{R}$ is the relative position vector, and $\mathbf{v}$ is the relative velocity of the two particles.

7.12 The observed period of the binary system Cygnus X-1, presumed to be a bright star and a black hole, is 5.6 days. If the mass of the visible component is $20\,M_\odot$ and the black hole has a mass of $16\,M_\odot$, show that the semimajor axis of the orbit of the black hole relative to the visible star is roughly one-fifth the distance from Earth to the Sun.

7.13 **(a)** Using the coordinate convention given is Section 7.4 for the restricted three-body problem, find the coordinates (x', y') of the two Lagrangian points, L_4 and L_5.
(b) Show that the gradient of the effective potential function $V(x', y')$ vanishes at L_4 and L_5.

7.14 A proton of mass m_p with initial velocity $\mathbf{v}_0$ collides with a helium atom, mass $4m_p$, that is initially at rest. If the proton leaves the point of impact at an angle of $45°$ with its original line of motion, find the final velocities of each particle. Assume that the collision is perfectly elastic.

7.15 Work Problem 7.14 for the case that the collision is inelastic and that Q is equal to one-fourth of the initial energy of the proton.

7.16 Referring to Problem 7.14, find the scattering angle of the proton in the center of mass system.

7.17 Find the scattering angle of the proton in the center-of-mass system for Problem 7.15.

7.18 A particle of mass m with initial momentum p_1 collides with a particle of equal mass at rest. If the magnitudes of the final momenta of the two particles are p_1' and p_2', respectively, show that the energy loss of the collision is given by

$$Q = \frac{p_1' p_2'}{m} \cos\psi$$

where ψ is the angle between the paths of the two particles after colliding.

7.19 A particle of mass m_1 with an initial kinetic energy T_1 makes an elastic collision with a particle of mass m_2 initially at rest. m_1 is deflected from its original direction with a kinetic energy T_1' through an angle ϕ_1 as in Figure 7.6.1. Letting $\alpha = m_2/m_1$ and $\gamma = \cos\phi_1$, show that the fractional kinetic energy lost by m_1, $\Delta T_1/T_1 = (T_1 - T_1')/T_1$, is given by

$$\frac{\Delta T_1}{T_1} = \frac{2}{1+\alpha} - \frac{2\gamma}{(1+\alpha)^2}\left(\gamma + \sqrt{\alpha^2 + \gamma^2 - 1}\right)$$

7.20 Derive Equation 7.6.18.

7.21 A particle of mass m_1 scatters elastically from a particle of mass m_2 initially at rest as described in Problem 7.19. Find the curve $r(\phi_1)$ such that the time it takes the scattered particle to travel from the collision point to any point along the curve is a constant.

7.22 A uniform chain lies in a heap on a table. If one end is raised vertically with uniform velocity v, show that the upward force that must be exerted on the end of the chain is equal to the weight of a length $z + (v^2/g)$ of the chain, where z is the length that has been uncoiled at any instant.

7.23 Find the differential equation of motion of a raindrop falling through a mist collecting mass as it falls. Assume that the drop remains spherical and that the rate of accretion is proportional to the cross-sectional area of the drop multiplied by the speed of fall. Show that if the drop starts from rest when it is infinitely small, then the acceleration is constant and equal to $g/7$.

7.24 A uniform heavy chain of length a hangs initially with a part of length b hanging over the edge of a table. The remaining part, of length $a - b$, is coiled up at the edge of the table. If the chain is released, show that the speed of the chain when the last link leaves the end of the table is $[2g(a^3 - b^3)/3a^2]^{1/2}$.

7.25 A balloon of mass M containing a bag of sand of mass m_0 is filled with hot air until it becomes buoyant enough to rise ever so slightly above the ground, where it then hovers in equilibrium. Sand is then released at a constant rate such that all of it is dumped out in a time t_0. Find (**a**) the height of the balloon and (**b**) its velocity when all the sand has been released. Assume that the upward buoyancy force remains constant and neglect air resistance. (**c**) Assume that $\epsilon = m_0/M$ is very small, and find a power series expansion of your solutions for parts (a) and (b) in terms of this ratio. (**d**) Letting $M = 500$ kg, $m_0 = 10$ kg, and $t_0 = 100$ s, and keeping only the first-order term in the expansions obtained in part (c), find a numerical value for the height and velocity attained when all the sand has been released.

7.26 A rocket, whose total mass is m_0, contains a quantity of fuel, whose mass is ϵm_0 $(0 < \epsilon < 1)$. Suppose that, on ignition, the fuel is burned at a constant mass-rate k, ejecting gasses with a constant speed V relative to the rocket. Assume that the rocket is in a force-free environment.
(**a**) Find the distance that the rocket has traveled at the moment it has burnt all the fuel.
(**b**) What is the maximum possible distance that the rocket can travel during the burning phase? Assume that it starts from rest.

7.27 A rocket traveling through the atmosphere experiences a linear air resistance $-k\mathbf{v}$. Find the differential equation of motion when all other external forces are negligible. Integrate the equation and show that if the rocket starts from rest, the final speed is given by

$$v = V\alpha[1 - (m/m_0)^{1/\alpha}]$$

where V is the relative speed of the exhaust fuel, $\alpha = |\dot{m}/k| =$ constant, m_0 is the initial mass of the rocket plus fuel, and m is the final mass of the rocket.

7.28 Find the equation of motion for a rocket fired vertically upward, assuming g is constant. Find the ratio of fuel to payload to achieve a final speed equal to the escape speed v_e from the Earth if the speed of the exhaust gas is kv_e, where k is a given constant, and the fuel burning rate is $|\dot{m}|$. Compute the numerical value of the fuel–payload ratio for $k = \frac{1}{4}$, and $|\dot{m}|$ equal to 1% of the mass of the fuel per second.

7.29 Alpha Centauri is the nearest star system, about 4 light years from Earth. Assume that an ion rocket has been built to travel to Alpha Centauri. Suppose the exhaust velocity of the ions is one-tenth the speed of light. Let the initial mass of the fuel be twice that of the payload (ignore the mass of the rocket, itself). Also, assume that it takes about 100 hours to exhaust all the fuel of the rocket. How long does it take the rocket to reach Alpha Centauri? (The speeds are small enough that you can neglect the effects of special relativity.)

7.30 Consider the ion rocket described in Problem 7.29. Let's compare it to a chemical rocket whose exhaust velocity is 3 km/s. In the case of the ion rocket, 1 kg of fuel accelerates 1 kg of payload to a final velocity v_f. What fuel mass is required to accelerate the same payload to the same final velocity with the chemical rocket? (In each case, ignore the mass of the rocket.)

Computer Problems

C 7.1 Let two particles ($m_1 = m_2 = 1$ kg) repel each other with equal and opposite forces given by

$$\mathbf{F}_{12} = k\frac{b^2}{r^2}\mathbf{r}_{12} = -\mathbf{F}_{21}$$

7.1

Center of mass

$$\vec{r}_{cm} = \frac{1}{M} \sum m_i \vec{r}_i$$

$$M = 3$$

$$\vec{r}_{cm} = \frac{1}{3}\left((\hat{x}+\hat{y}) + (\hat{y}+\hat{z}) + \hat{z}\right)$$

$$= \frac{1}{3}(\hat{x} + 2\hat{y} + 2\hat{z})$$

Velocity of CM

$$\vec{v}_{cm} = \dot{\vec{r}}_{cm} = \frac{1}{M} \sum m_i \vec{v}_i$$

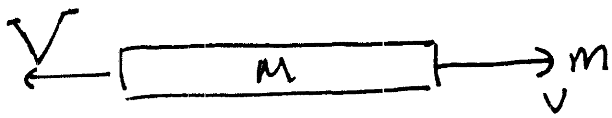
$$= \frac{1}{3}(2\hat{x} + \hat{y} + \hat{x} + \hat{y} + \hat{z})$$

$$= \frac{1}{3}(3\hat{x} + 2\hat{y} + \hat{z})$$

Momentum of CM

$$\vec{p}_{cm} = M\vec{v}_{cm} = 3\vec{v}_{cm} = 3\hat{x} + 2\hat{y} + \hat{z}$$

(7.3)



Muzzle velocity $v_0 = v + V$

velocity relative to ground

Velocity of bullet v

<u>Conserve Momentum</u>

$$0 = mv - MV$$

$$v = \frac{M}{m} V \qquad V = \gamma v$$

$$\gamma = \frac{m}{M}$$

$$v = v_0 - V = v_0 - \gamma v$$

$$(1+\gamma) v = v_0$$

$$v = v_0 / (1+\gamma)$$ Velocity of bullet

Recoil Velocity

$$V = \gamma v = \frac{\gamma v_0}{1+\gamma}$$

(7.8) Let $v = v_2 - v_1$ $\quad m = m_1 + m_2$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

See if $T = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} \mu v^2$ can be converted into this.

$$v_{cm} = \frac{P_{cm}}{M} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

$$T = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} \mu v^2$$

$$= \frac{1}{2}(m_1 + m_2)\left(\frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}\right)^2 + \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)(v_2 - v_1)^2$$

$$= \frac{1}{2(m_1+m_2)}\Big[m_1^2 v_1^2 + m_2^2 v_2^2 + 2 m_1 m_2 v_1 v_2 + m_1 m_2 v_2^2 + m_1 m_2 v_1^2 + 2 m_1 m_2 v_1 v_2\Big]$$

$$= \frac{1}{2 m_1 + m_2}\Big[m_1(m_1 + m_2) v_1^2 + m_2(m_1 + m_2) v_2^2\Big]$$

$$= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \quad \checkmark$$

(7.9) Since there are no external forces, the momentum and velocity of the center-of-mass are conserved.

$$T_i = T_f + Q$$

$$Q = T_i - T_f = \frac{1}{2} M v_{cm}^2 + \frac{1}{2}\mu v^{I2} - \frac{1}{2} M v_{cm}^2 - \frac{1}{2}\mu v^{F2}$$

using result of the previous problem.

$$Q = \frac{1}{2}\mu\left(v^{I2} - v^{F2}\right) = \frac{1}{2}\mu v^{I2}\left(1 - \frac{v^{F2}}{v^{I2}}\right)$$

$$= \frac{1}{2}\mu v^2\left(1 - \varepsilon^2\right)$$

$$v^F \equiv V_2^F - V_1^F \qquad v^I \equiv v \equiv V_2^I - V_1^I$$

$$\varepsilon \equiv \frac{v^F}{v^I}$$

7.12

$\tau = 5.6$ days $\qquad M_v = 20 M_\odot$

$M_B = 16 M_\odot \qquad \tau = 4.8\times10^5\,s$

$M_\odot = 2\times10^{30}\,kg$

$$\tau = 2\pi\left[G(M_v + M_B)\right]^{-1/2} a^{3/2}$$

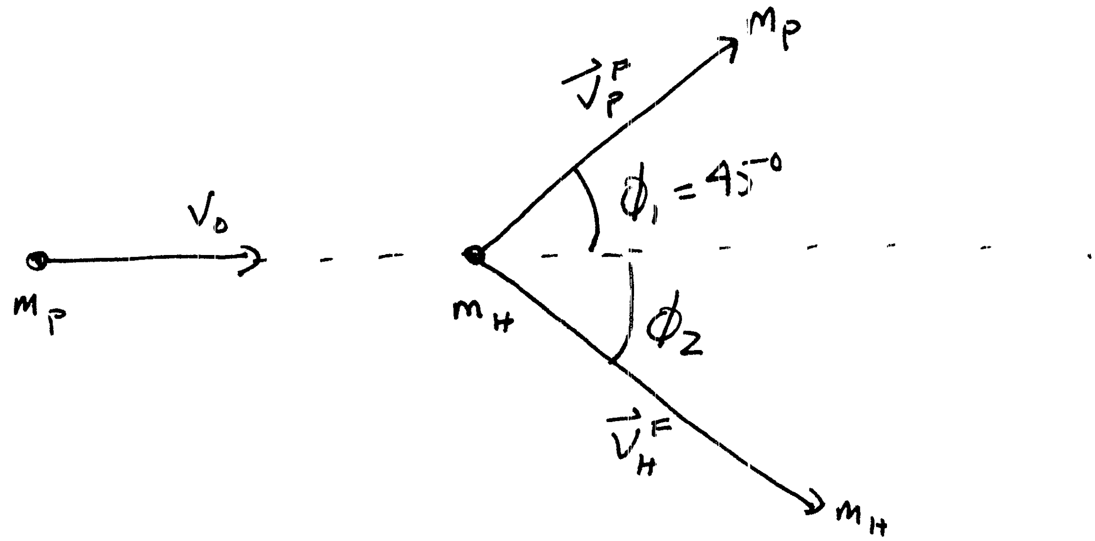
$$a = \left[\frac{\tau}{2\pi}\sqrt{G(M_v + M_B)}\right]^{2/3}$$

$$= \left[\frac{4.8\times10^5 s}{2\pi}\sqrt{\left(6.67\times10^{-11}\frac{Nm^2}{kg^2}\right)(36)\left(2\times10^{30}kg\right)}\right]^{2/3}$$

$$= 3\times10^{10}\,m = \frac{1}{5}\,AU$$

1 AU = 1 radius of earth orbit $= 1.5\times10^{11}\,m$

7.14



Collision is elastic, $Q = 0$.

Momentum $\quad \vec{P}_p^I = \vec{P}_p + \vec{P}_H^F$

x-component

(1) $\quad m_p V_0 = m_p V_p^F \cos 45^\circ + 4 m_p V_H^F \cos \phi_2$

y-component

(2) $\quad 0 = m_p V_p^F \sin 45^\circ - 4 m_p V_H^F \sin \phi_2$

Energy

(3) $\quad \frac{1}{2} m_p V_0^2 = \frac{1}{2} m_p V_p^{F2} + \frac{1}{2}(4 m_p) V_H^{F2}$

Solve (3) for V_H^F

$$V_H^F = \frac{V_P^F \sin 45}{4 \sin \phi_2} = \frac{V_P^F}{4\sqrt{2} \sin \phi_2} \quad (4)$$

Simplify (1)

$$V_0 - \frac{V_P^F}{\sqrt{2}} = 4 V_H^F \cos \phi_2$$

Simplify (2)

$$\frac{V_P^F}{\sqrt{2}} = 4 V_H^F \sin \phi_2$$

$(1)^2 + (2)^2$

$$\left(V_0 - \frac{V_P^F}{\sqrt{2}} \right)^2 + \frac{{V_P^F}^2}{2} = 16 {V_H^F}^2$$

$$V_0^2 - \sqrt{2}\, V_0 V_P^F + {V_P^F}^2 = 16 {V_H^F}^2 \quad (5)$$

Multiply (3) by 4

$$4v_0^2 = 4v_P^{F2} + 16v_H^{F2} \qquad (6)$$

(5) - (6)

$$-3v_0^2 - \sqrt{2}\,v_0 v_P^F + 5v_P^{F2} = 0 \qquad (7)$$

Quadratic Formula

$$v_P^F = \frac{\sqrt{2}\,v_0 \pm \sqrt{2v_0^2 + 60v_0^2}}{10}$$

$$= \frac{v_0}{10}\left(\sqrt{2} + \sqrt{62}\right) = 0.929\,v_0$$

(2) / (1)

$$\tan\phi_2 = \frac{v_P^F/\sqrt{2}}{v_0 - v_P^F/\sqrt{2}} = \frac{1}{\frac{\sqrt{2}\,v_0}{v_P^F} - 1} = 1.91$$

$$\phi_2 = 62^\circ$$

Substitute into (4)

$$v_H^F = \frac{v_p^F}{4\sqrt{2}\sin\phi_2} = \frac{v_0(0.9288)}{4\sqrt{2}\sin 62}$$

$$= 0.185 v_0$$

In component form

$$\vec{v}_H^F = (0.185 v_0 \cos 62, -0.185 v_0 \sin 62, 0)$$

$$\vec{v}_p^F = (0.923 v_0 \cos 45, 0.923 v_0 \sin 45, 0)$$

7.16 $\quad$ θ angle in CM

$$\tan\phi_1 = \tan 45 = 1 = \frac{\sin\theta}{\gamma + \cos\theta}$$

Since elastic $\gamma = \frac{m_1}{m_2} = \frac{1}{4}$

$$1 = \frac{\sin\theta}{\frac{1}{4} + \cos\theta}$$

$$\left(\frac{1}{4} + \cos\theta\right)^2 = \sin^2\theta = 1 - \cos^2\theta$$

$$\frac{1}{16} + \frac{1}{2}\cos\theta + \cos^2\theta = 1 - \cos^2\theta$$

$$2\cos^2\theta + \frac{1}{2}\cos\theta - \frac{15}{16} = 0$$

$$\cos\theta = \frac{-\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{15}{2}}}{4}$$

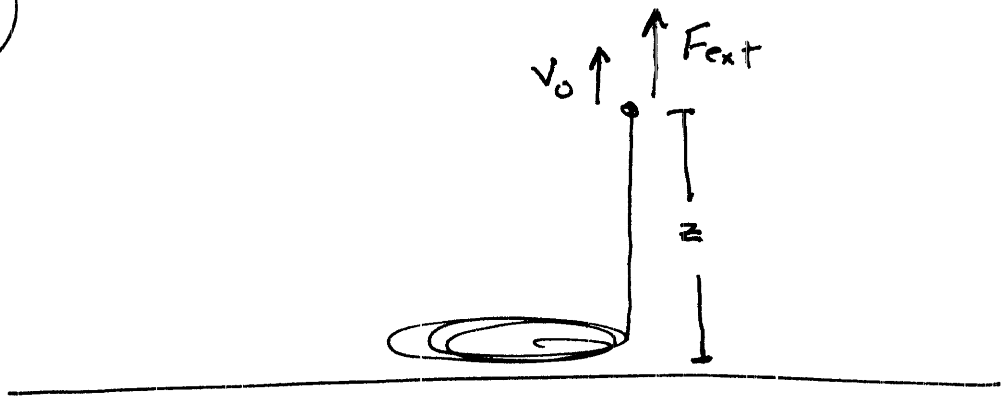
$$\cos\theta = \frac{-1 \pm \sqrt{1+30}}{8}$$

$$= \frac{-1 \pm \sqrt{31}}{8}$$

Take positive root

$$\theta = \cancel{55}\ 55,2$$

7.22



Let upward be positive.

Newton II

$$F_{ext} - m(t)g = m(t)\dot{\vec{v}}_{cm} - v_{rel}\dot{m}$$

$$v_{rel} = v_{env} - v_{sys} = 0 - v_0 = -v_0$$

$$\dot{v}_{cm} = 0 \quad \text{since } v_0 \text{ constant}$$

$$F_{ext} = m(t)g + \dot{m}v_0$$

Note, chain provides force in negative direction

$$m(t) = \lambda z$$

where z is the height of the end and λ is the mass density.

$$\dot{m}(t) = \lambda\dot{z} = \lambda v_0$$

$$F_{ext} = \lambda z g + \lambda v_0^2$$

$$= \lambda g \underbrace{\left(z + v_0^2/g \right)}_{\text{effective length of chain}}$$

(7.26) $\vec{F}_{ext} + \dot{m}\vec{v}_{rel} = m\dot{\vec{v}}_{cm}$

Force free environment, $F_{ext} = 0$. Constant relative velocity of exhausted fuel

$$|\vec{v}_{rel}| = \mathcal{V}, \quad v_{rel} = -\mathcal{V}$$

Constant rate of burning, $\dot{m} = -k$.

$$m(t) = m_0 - kt$$

$$m(t_f) = m_0 - kt_f = m_0 - \varepsilon m_0$$

$$t_f = \frac{\varepsilon m_0}{k}$$

$$\dot{m}\, v_{rel} = m\dot{v}$$

$$(-k)(-\mathcal{V}) = m(t)\, \dot{v} = (m_0 - kt)\dot{v}$$

$$\frac{dv}{dt} = \frac{k\mathcal{V}}{m_0 - kt}$$

$$\int_0^{v_0} dv = \int_0^t \frac{k\mathcal{V}\, dt}{m_0 - kt}$$

$$V = -\frac{1}{k} k \mathcal{V} \ln(m_0 - kt)\Big|_0^t$$

$$= -\mathcal{V} \ln\left(\frac{m_0 - kt}{m_0}\right)$$

$$\frac{dx}{dt} = -\mathcal{V} \ln(1 - \gamma t) \qquad \gamma = \frac{k}{m_0}$$

$$\int_0^x dx = -\mathcal{V} \int_0^t \ln(1-\gamma t)\, dt$$

Let $u = 1-\gamma t \quad du = -\gamma dt$

$$x = \frac{\mathcal{V}}{\gamma} \int_1^{1-\gamma t} \ln u \, du$$

$$= \frac{\mathcal{V}}{\gamma} \left[u \ln u - u \right]_1^{1-\gamma t}$$

$$= \frac{\mathcal{V}}{\gamma} \Big[(1-\gamma t) \ln(1-\gamma t) - (1-\gamma t)$$
$$- (1 \ln(1) - 1) \Big]$$

$$x = \frac{v}{\gamma}\left[(1-\gamma t)\ln(1-\gamma t) + \gamma t\right]$$

$$x = vt + \frac{v}{\gamma}(1-\gamma t)\ln(1-\gamma t)$$

The rocket burns all it fuel at $t_f = \dfrac{\varepsilon m_0}{k}$

$$\gamma t_f = \left(\frac{k}{m_0}\right)\left(\frac{\varepsilon m_0}{k}\right) = \varepsilon$$

$$x_f = \frac{v \varepsilon m_0}{k} + \frac{v m_0}{k}(1-\varepsilon)\ln(1-\varepsilon)$$

(a) $$x_f = \frac{v m_0}{k}\underbrace{\left[\varepsilon + (1-\varepsilon)\ln(1-\varepsilon)\right]}_{f(\varepsilon)}$$

(b)

$$\frac{df}{d\varepsilon} = -\ln(1-\varepsilon) \neq 0$$, no minimum.

Max is achieved at $\varepsilon = 1$

$$x_{max} = \frac{v m_0}{k}$$