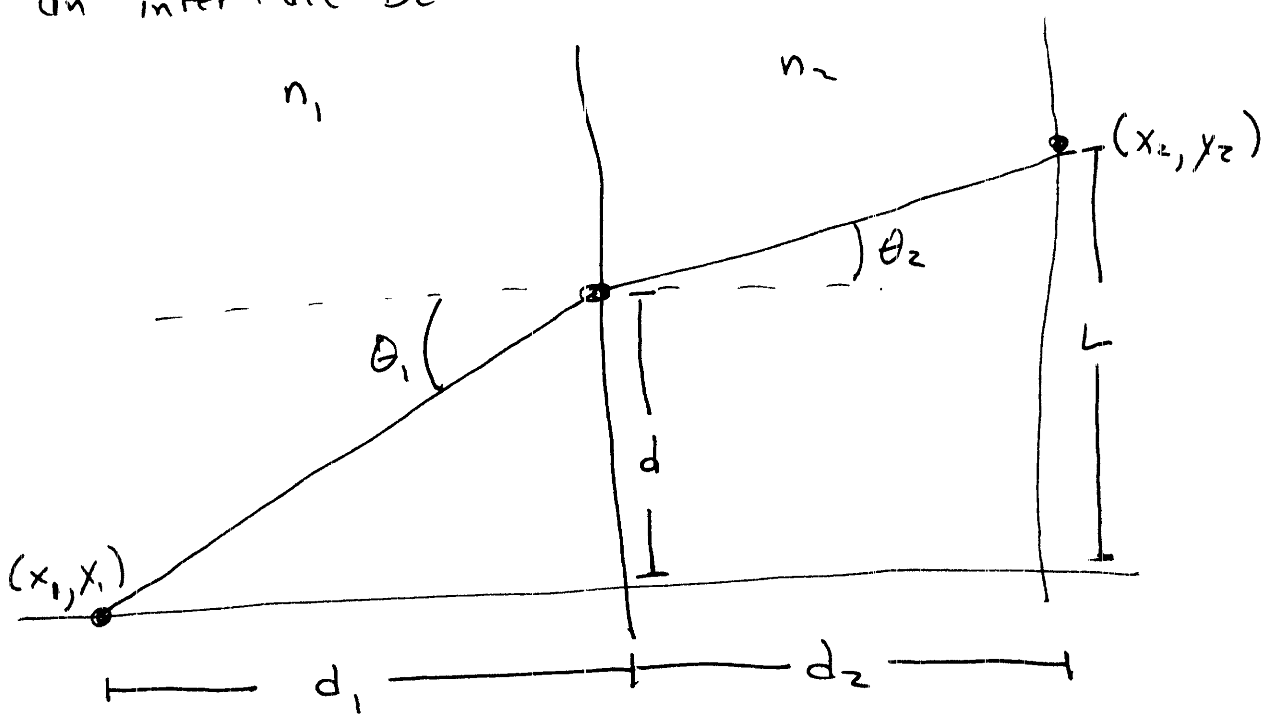
Variational Methods

Variational methods use the calculus of variations to describe a physical property as a extrema along some path.

Example Fermat's Principle - The path that light takes between two points minimize the time required.

$$\Rightarrow \quad n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Proof Select two points on different sides of an interface between two materials



The path taken is evidently straight in both media, so the question is where does the path strike the interface.

The speed of light in the two medium is

$$c_1 = \frac{c}{n_1} \qquad c_2 = \frac{c}{n_2}$$

where n_i is the index of refraction.

The distance traveled in medium 1 is $\ell_1 = \sqrt{d^2 + d_1^2}$

and the distance traveled in medium 2 is $\ell_2 = \sqrt{d_2^2 + (L-d)^2}$

The total time to travel between the two points is

$$T = \frac{\ell_1}{c_1} + \frac{\ell_2}{c_2} = \frac{1}{c}\left(n_1 \ell_1 + n_2 \ell_2\right)$$

$$= \frac{1}{c}\left(n_1 \sqrt{d^2 + d_1^2} + n_2 \sqrt{d_2^2 + (L-d)^2}\right)$$

Find minimum value,

$$\frac{dT}{dd} = 0 = \frac{1}{c}\left[\frac{n_1 d}{\sqrt{d^2 + d_1^2}} - \frac{n_2 (L-d)}{\sqrt{d_2^2 + (L-d)^2}}\right] = 0$$

$$\frac{d}{\sqrt{d^2+d_1^2}} = \sin\theta_1 \qquad \frac{(L-d)}{\sqrt{d_2^2+(L-d)^2}} = \sin\theta_2$$

$$\Rightarrow \quad n_1 \sin\theta_1 = n_2 \sin\theta_2$$

So the path selected out of all the possible paths minimizes the travel time between the two points.

<u>Newton's Law</u> - Momentum is conserved in all pairwise interactions

$$\frac{d\vec{p}_{12}}{dt} + \frac{d\vec{p}_{21}}{dt} = 0$$

is equivalent to

<u>Hamilton's Principle</u> - The motion of the system from t_1 to t_2 is such that the action, S, has a stationary value.

$\Rightarrow$ The principle of stationary action

Action (S)

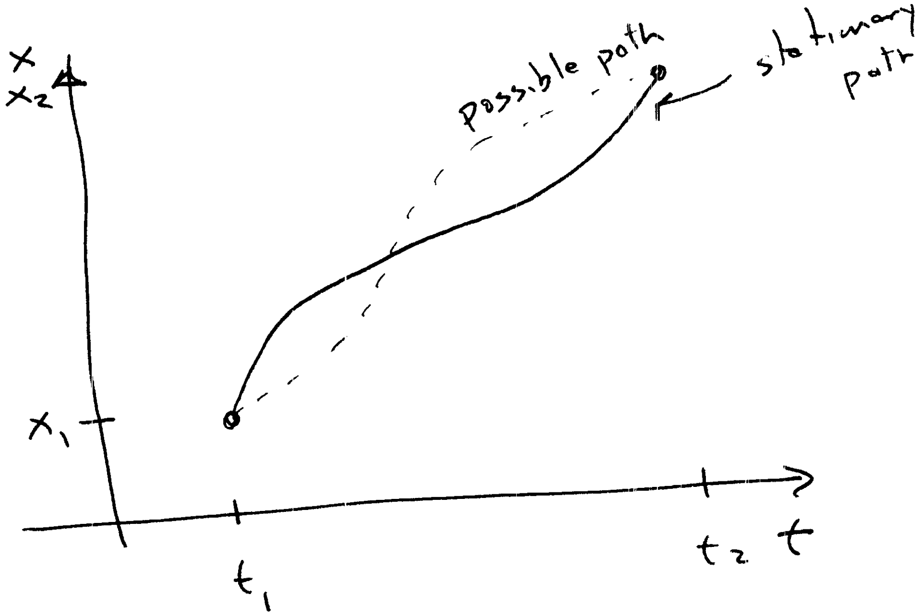
$$S = \int_{t_1}^{t_2} L \, dt$$

Lagrangian (L)

$$L = T - V$$

T ← kinetic energy

The trajectory taken is the one the leaves the action stationary $\delta S = 0$ with endpoints fixed



The Lagrangian is a function of x, $\dot{x}$, and t. As we vary the path $\delta x(t)$ only the parts that depend on the path change

$$\delta L = \frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta \dot{x}$$

$$\delta \dot{x} = \frac{d}{dt}\delta x$$

$$\delta L = \frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\frac{d}{dt}\delta x$$

$$\delta S = \int_{t_1}^{t_1} \delta L\, dt = \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\frac{d}{dt}\delta x\right) dt$$

$$= 0$$

Integrate the second term by parts

$$\delta S = \underbrace{\frac{\partial L}{\partial \dot{x}_i}\delta x \Big|_{t_1}^{t_2}}_{\text{0 because endpoints not varied.}} + \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right)\delta x\, dt$$

The action is stationary if

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$$

Let's try it out

Simple Harmonic Oscillator

$$T = \frac{1}{2} m \dot{x}^2 \qquad V = \frac{1}{2} k x^2$$

$$L = T - V = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

$$\frac{\partial L}{\partial x} = -2kx \qquad \frac{\partial L}{\partial \dot{x}} = 2m\dot{x}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 2m\ddot{x}$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = -2kx - 2m\ddot{x} = 0$$

$$m\ddot{x} + kx = 0 \quad \checkmark$$

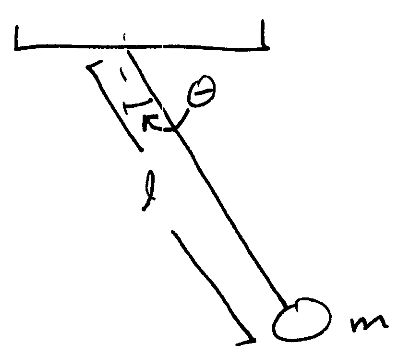
Now the magic

Lagrange's equations work for any coordinates q_i under constraints of the form $f(\{q_i\}, t) = 0$ called <u>holonomic</u> constraints.

If we N <u>generalized coordinates</u> $\{q_i\}$ then the equations of motion are given by

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$$

<u>Ex</u> <u>Simple Pendulum</u>



$$T = \frac{1}{2} m (l\dot{\theta})^2 \qquad V = mgl(1-\cos\theta)$$

$$L = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1-\cos\theta)$$

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

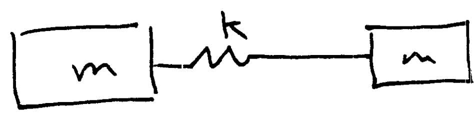
$$\frac{\partial L}{\partial \dot\theta} = ml^2\dot\theta \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = ml^2\ddot\theta$$

<u>Lagrange's Eqn</u>

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = 0 = -mgl\sin\theta - ml^2\ddot\theta$$

$$\ddot\theta + \frac{g}{l}\sin\theta = 0$$

<u>Example</u> Two blocks of mass m on a frictionless surface connected by spring



Let x_1 be center of the first mass, x_2 center of the second mass, and d d_0 the separation when the spring is at equilibrium.

$$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$$

$$V = \frac{1}{2} k \left(x_2 - x_1 - d_0 \right)^2$$

$$L = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 - \frac{1}{2} k \left(x_2 - x_1 - d_0 \right)^2$$

$$\frac{\partial L}{\partial x_1} = k \left(x_2 - x_1 - d_0 \right) \qquad \frac{\partial L}{\partial x_2} = -k \left(x_2 - x_1 - d_0 \right)$$

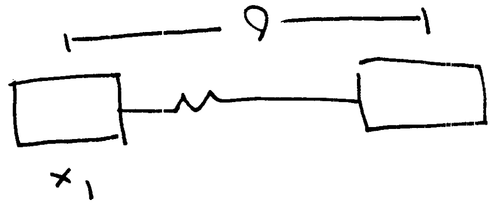
$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_1} = m \ddot{x}_1 \qquad \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_2} = m \ddot{x}_2$$

<u>Lagrange's Eqns</u>

$$\frac{\partial L}{\partial x_1} - \frac{d}{dt} \frac{\partial L}{\partial x_1} = k \left(x_2 - x_1 - d_0 \right) - m \ddot{x}_1 = 0$$

$$\frac{\partial L}{\partial x_2} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_2} = -k \left(x_2 - x_1 - d_0 \right) - m \ddot{x}_2 = 0$$

But what if I don't like those variables?



$$x_2 = x_1 + l \qquad \dot{x}_2 = \dot{x}_1 + \dot{l}$$

$$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m (\dot{x}_1 + \dot{l})^2$$

$$V = \frac{1}{2} k (l - l_0)^2$$

$$L = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m (\dot{x}_1 + \dot{l})^2 - \frac{1}{2} k (l - l_0)^2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} = m\ddot{x}_1 + m(\ddot{x}_1 + \ddot{l}) \qquad \frac{\partial L}{\partial x_1} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{l}} = m(\ddot{x}_1 + \ddot{l}) \qquad \frac{\partial L}{\partial l} = -k(l - l_0)$$

EOM

x_1: $\quad 2\ddot{x}_1 + \ddot{l} = 0$

l: $\quad -k(l - l_0) - m(\ddot{x}_1 + \ddot{l}) = 0$

$$\ddot{x}_1 = -\frac{\ddot{l}}{2}$$

$$\frac{m}{2}\ddot{l} + k(l - l_0) = 0$$

$$\ddot{l} + \frac{2k}{m}(l - l_0) = 0$$

$$\omega_0^2 = \frac{2k}{m}$$

Or how about x_{cm}, l ?

$$x_{cm} = \frac{x_1 + x_2}{2} \qquad l = x_2 - x_1 \cancel{+l_0}$$

$$x_1 = x_{cm} - l/2 \qquad x_2 = x_{cm} + l/2$$

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2 = \frac{1}{2}m\left(\dot{x}_{cm} - \frac{\dot{l}}{2}\right)^2 + \frac{1}{2}m\left(\dot{x}_{cm} + \frac{\dot{l}}{2}\right)^2$$

$$= \frac{1}{2}m\left(2\dot{x}_{cm}^2 + \frac{\dot{l}^2}{2}\right)$$

$$T = m\dot{x}_{cm}^2 + \frac{1}{4}m\dot{l}^2$$

$$V = \frac{1}{2}k(l-l_0)^2$$

$$L = T - V = m\dot{x}_{cm}^2 + \frac{1}{4}m\dot{l}^2 - \frac{1}{2}k(l-l_0)^2$$

$$\frac{\partial L}{\partial x_{cm}} = 0 \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{cm}} = 2m\ddot{x}_{cm}$$

$$\frac{\partial L}{\partial l} = -k(l-l_0) \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{l}} = 2m\ddot{l}$$

EOM

x_{cm}: $\quad 2m\ddot{x}_{cm} = 0 \qquad \dot{x}_{cm} = \text{constant}$

l: $\quad -k(l-l_0) - 2m\ddot{l} = 0$

$$\ddot{l} + \frac{k}{2m}(l-l_0) = 0$$