Lecture 2 - Lagrangians

Something cool happened in last example. We wound up with an EOM $2m\ddot{x}_{cm} = 0$, which meant the quantity $\dot{x}_{cm}$ was constant. This occured because the ~~potential and~~ Lagrangian did not depend on that coordinate.

<u>Ignorable Coordinate</u> - Any generalized coordinate q such that the Lagrangian is independent of q, but not necessarily $\dot{q}$.

<u>Generalized Momenta</u>

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

conjugate to q_i

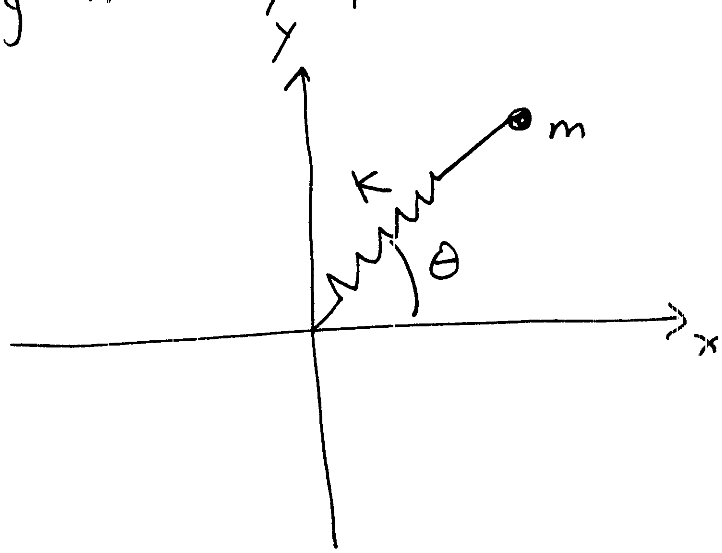
<u>Ex</u> For free particle, $T = \frac{1}{2} m \dot{x}^2 = L$

$$p = \frac{\partial T}{\partial \dot{x}} = m\dot{x} \quad \checkmark$$

The generalized momenta conjugate to an ignorable coordinate is conserved.

$$\dot{p}_i = 0 \qquad p_i = \text{constant}$$

<u>Example</u> Isotropic harmonic oscillator moving in x-y plane.



$$x = l \cos\theta \qquad y = l \sin\theta$$

$$\dot{x} = \dot{l}\cos\theta - l\sin\theta\,\dot{\theta}$$

$$\dot{y} = \dot{l}\sin\theta + l\cos\theta\,\dot{\theta}$$

$$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right)$$

$$T = \frac{1}{2}m\Big(\dot{l}^2\cos\theta - 2l\dot{l}\cos\theta\sin\theta\dot{\theta} + l^2\sin^2\theta\dot{\theta}^2$$

$$+ \dot{l}^2\sin\theta + 2l\dot{l}\cos\theta\sin\theta\dot{\theta} + l^2\cos^2\theta\dot{\theta}^2 \Big)$$

$$= \frac{1}{2}m\dot{l}^2 + \frac{1}{2}m(l\dot{\theta})^2$$

$$V = \frac{1}{2}kl^2$$

$$L = \frac{1}{2}m\dot{l}^2 + \frac{1}{2}m(l\dot{\theta})^2 - \frac{1}{2}kl^2$$

The coordinate θ does not appear in L,

$\Rightarrow$ p_θ is conserved.

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} = \text{constant}$$

angular momentum $I\dot{\theta}$

$$\frac{dp_\theta}{dt} = 0$$

$$0 = \frac{\partial L}{\partial \ell} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\ell}}$$

$$= -k\ell + m\ell\dot{\theta}^2 - m\ddot{\ell}$$

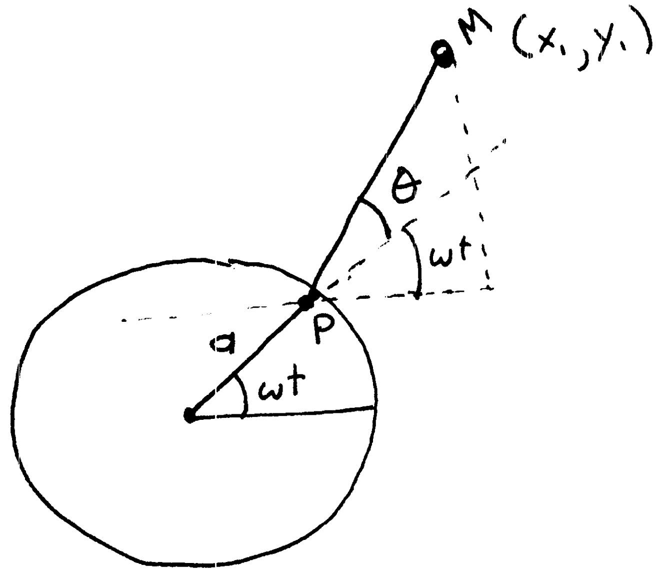
$$P_\theta = \text{const} \equiv A = m\ell^2\dot{\theta}$$

$$\dot{\theta} = \frac{A}{m\ell^2}$$

$$0 = -k\ell + m\ell\left(\frac{A}{m\ell^2}\right) - m\ddot{\ell}$$

$$\ddot{\ell} + \frac{k}{m}\ell - \frac{A}{m\ell} = 0$$

Ex Mass attached to disk rotating at angular velocity ω with massless rod of length l.



No gravity

$$x_p = a\cos\omega t \qquad y_p = a\sin\omega t$$

$$x_1 = x_p + l\cos(\theta+\omega t) \qquad y_1 = y_p + l\sin(\theta+\omega t)$$

$$L = T - V = T$$

$$T = \tfrac{1}{2}m\left(\dot{x}_1^2 + \dot{y}_1^2\right)$$

$$\dot{x}_1 = -a\omega\sin\omega t - l\sin(\theta+\omega t)(\dot{\theta}+\omega)$$

$$\dot{y}_1 = a\omega\cos\omega t + l\cos(\theta+\omega t)(\dot{\theta}+\omega)$$

$$\frac{2T}{m} = \left(-a\omega \sin\omega t - l\sin(\theta+\omega t)(\dot\theta+\omega)\right)^2$$

$$+\left(a\omega\cos\omega t + l\cos(\theta+\omega t)(\dot\theta+\omega)\right)^2$$

$$= a^2\omega^2 + l^2(\dot\theta+\omega)^2 + 2\omega a l \sin\omega t \sin(\theta+\omega t)(\dot\theta+\omega)$$

$$+ 2al\omega\cos\omega t\cos(\theta+\omega t)(\dot\theta+\omega)$$

$$\cancel{\frac{2}{m}\frac{\partial L}{\partial\theta} = +2\omega a l}$$

Look at last two terms

$$2al\omega(\dot\theta+\omega)\left(\sin\omega t\sin(\theta+\omega t) + \cos\omega t\cos(\theta+\omega t)\right)$$

From math handbook

$$\sin\omega t\sin(\theta+\omega t) = \tfrac{1}{2}\left(\cos\theta - \cos(\theta+2\omega t)\right)$$

$$\cos\omega t\cos(\theta+\omega t) = \tfrac{1}{2}\left(\cos\theta + \cos(\theta+2\omega t)\right)$$

So the last term reduces to

$$2al\omega\cos\theta(\dot\theta+\omega)$$

$$L = \frac{1}{2} m a^2 \omega^2 + \frac{1}{2} m l^2 (\dot{\theta} + \omega)^2 + m a l \omega \cos\theta \, (\dot{\theta} + \omega)$$

$$\frac{\partial L}{\partial \theta} = - m a l \omega \sin\theta \, (\dot{\theta} + \omega)$$

$$\frac{\partial L}{\partial \dot{\theta}} = \cancel{\frac{1}{2} m l^2} \; m l^2 (\dot{\theta} + \omega) + m a l \omega \cos\theta$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = m l^2 \ddot{\theta} - m a l \omega \sin\theta \, \dot{\theta}$$

EOM

$$\cancel{0 = \frac{\partial L}{\partial \theta} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = m l^2 \ddot{\theta} - m a l \omega \sin\theta \, \dot{\theta}}$$

$$0 = - m a l \omega \sin\theta (\dot{\theta} + \omega) - m l^2 \ddot{\theta} + m a l \omega \sin\theta \, \dot{\theta}$$

A few things from UPI (maybe).

Rotational Kinetic Energy

$$T = \frac{1}{2} I \dot{\theta}^2$$

Moment of Inertia (I) - Look up for now

Rod about center

$$I = \frac{m l^2}{12}$$ l = length

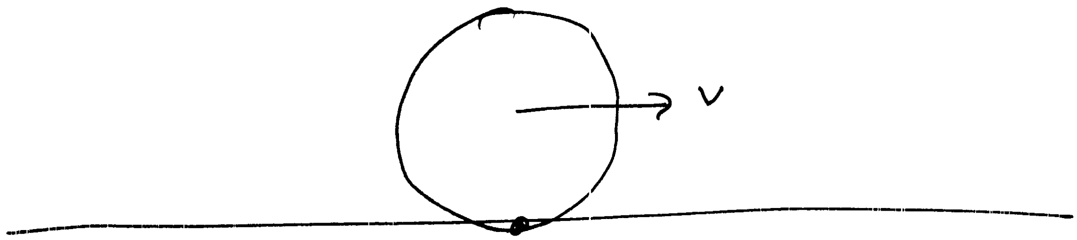
Rod about end

$$I = \frac{m l^2}{3}$$

Disk about center or cylinder about center axis

$$I = \frac{m r^2}{2}$$

Condition of Rolling - If a disk rolls without slipping $v = r\dot{\theta}$

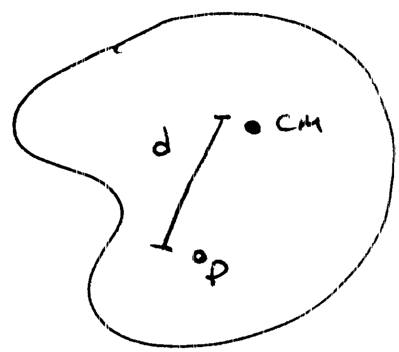


In time t, the disk moves $d = vt$. It turns through an angle θ. The length of perimeter rotated through is $\frac{\theta}{2\pi} \cdot 2\pi r = \theta r = d = vt$

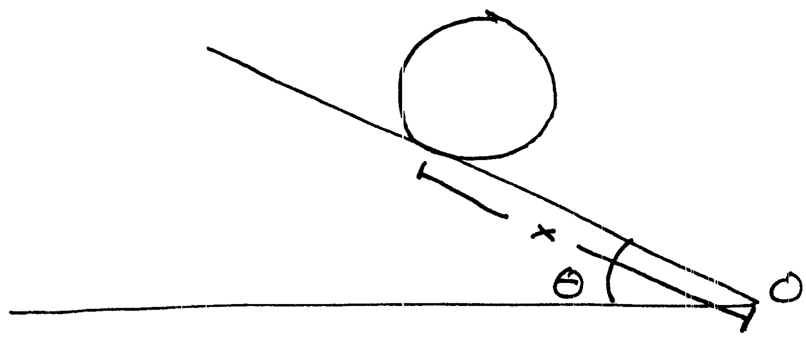
$$v = \dot{\theta} r$$

<u>Parallel-axis Thm</u>

$$I_p = I_{cm} + m d^2$$



Ex A sphere of radius a rolls down a rough surface.



$V(x) = +mgx\sin\theta$ $\qquad I$ = moment of inertia

$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\omega^2$

Condition of rolling $\qquad v = r\omega$

$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\left(\frac{v}{r}\right)^2$

$= \frac{1}{2}\left(m + \frac{I}{r^2}\right)\dot{x}^2 = \frac{1}{2}\left(m + \frac{I}{a^2}\right)\dot{x}^2$

$L = T - V = \frac{1}{2}\left(m + \frac{I}{a^2}\right)\dot{x}^2 - mgx\sin\theta$

$\frac{\partial L}{\partial x} = \cancel{mgx} - mg\sin\theta$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \left(m + \frac{I}{a^2}\right)\ddot{x}$

EOM

$$0 = \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = -mg\sin\theta - \left(m + \frac{I}{a^2}\right)\ddot{x} = 0$$

$$\ddot{x} = -\frac{mg\sin\theta}{m + I/a^2} = \frac{-g\sin\theta}{1 + \frac{I}{ma^2}} \equiv a$$

Look up moments

Sphere $\quad I = \frac{2}{5}ma^2 \quad \Rightarrow \quad a = \frac{-5}{7} g\sin\theta$

Thin Shell $\quad I = ma^2 \quad \Rightarrow \quad a = \frac{-1}{2} g\sin\theta$

Solid Cylinder $\quad I = \frac{ma^2}{2} \quad \Rightarrow \quad a = \frac{-2}{3} g\sin\theta$

Thin Disk $\quad I = \frac{ma^2}{2} \quad \Rightarrow \quad a = \frac{-2}{3} g\sin\theta$

Trajectory

$$x(t) = x_0 + v_0 t + \frac{1}{2}at^2$$

Hamiltonian

$$H = \underbrace{\sum_i \dot{q}_i p_i}_{\substack{mv^2 \\ \| \\ 2T}} - L$$

Hamiltonian

- $H = T + V$ but H is a function of q_i and p_i where L is a function of q_i and $\dot{q}_i$.

- Example of a <u>Legendre Transform</u>. In thermodynamics the total energy is $U(S, V, N)$ and the Helmholtz Free Energy

$$F = U - TS$$

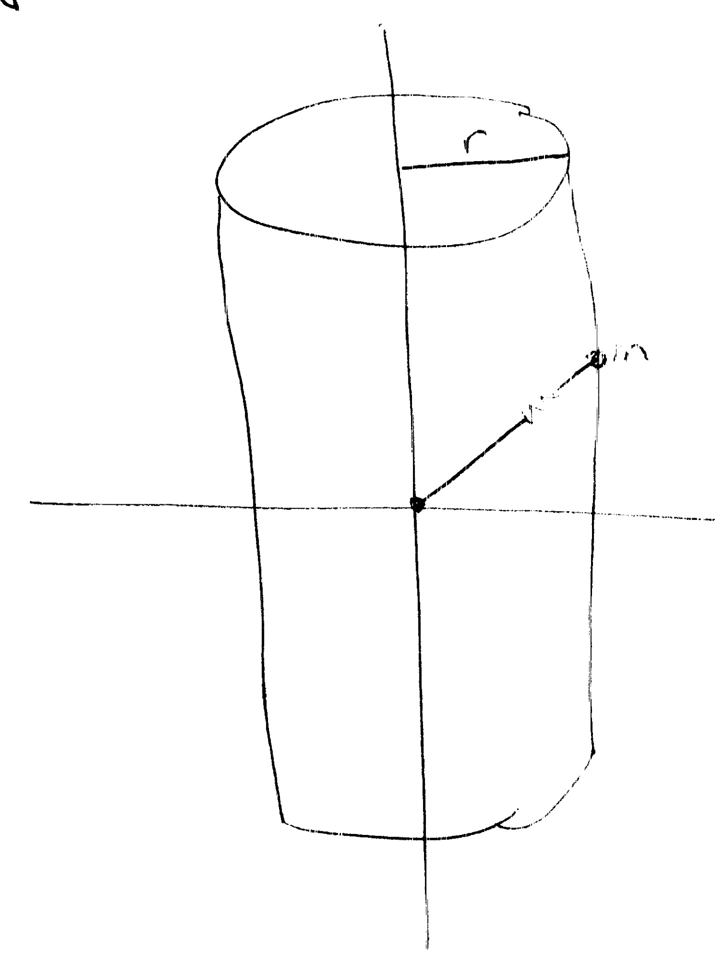
is a function of $F(T, V, N)$

By expanding δH and using Lagrange's equations we get Hamilton's Equations of Motion.

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \qquad \frac{\partial H}{\partial q_i} = -\dot{p}_i$$

Two first order differential equations instead of one second order.

Ex Particle of mass m confined to surface of a cylinder under a Hooke's Law restoring force.



In cylindrical coordinates,

$$T = \tfrac{1}{2}\vec{v}\cdot\vec{v} = \tfrac{1}{2}m\left(\dot{r}^2 + (r\dot{\theta})^2 + \dot{z}^2\right)$$

$$= \tfrac{1}{2}mr^2\dot{\theta}^2 + \tfrac{1}{2}m\dot{z}^2 \qquad \text{if } r \text{ constant}$$

$$V = \tfrac{1}{2}k\left(r^2 + z^2\right)$$

$$L = T - V = \tfrac{1}{2}mr^2\dot{\theta}^2 + \tfrac{1}{2}m\dot{z}^2 - \tfrac{1}{2}k\left(r^2 + z^2\right)$$

<u>Generalized Momenta</u>

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

<u>Hamiltonian</u> $\qquad H = T + V$

$$H = \tfrac{1}{2}mr^2\dot{\theta}^2 + \tfrac{1}{2}m\dot{z}^2 + \tfrac{1}{2}k\left(r^2 + z^2\right)$$

Hamiltonian must be written in terms of

$$\theta,\ p_\theta,\ z,\ p_z$$

$$\dot{\theta} = \frac{p_\theta}{mr^2} \qquad \dot{z} = \frac{p_z}{m}$$

$$H = \frac{1}{2}mr^2\left(\frac{p_\theta}{mr^2}\right)^2 + \frac{1}{2}m\left(\frac{p_z}{m}\right)^2 + \frac{1}{2}k\left(r^2+z^2\right)$$

$$= \frac{p_\theta^2}{2mr^2} + \frac{p_z^2}{2m} + \frac{1}{2}k\left(r^2+z^2\right)$$

<u>EOM</u>

$$\frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} = \dot{\theta} \qquad \frac{\partial H}{\partial \theta} = 0 = -\dot{p}_\theta$$

$\Rightarrow p_\theta$ conserved

$$\frac{\partial H}{\partial p_z} = \frac{p_z}{m} = \dot{z} \qquad \underbrace{\frac{\partial H}{\partial z} = kz = -\dot{p}_z}_{\text{Newton II}}$$