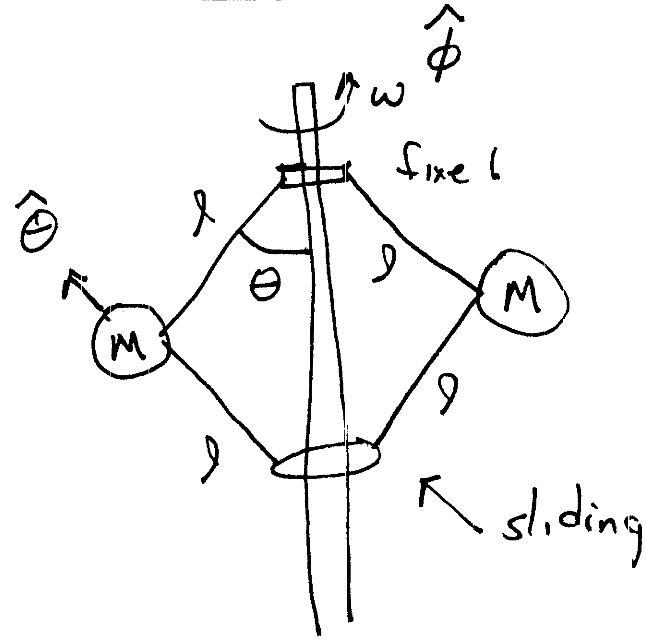
Flyball Governor

Invented James Watt



Mass of rods and supports negligible compaired to M.

$$V = -2Mgl\cos\Theta$$

Velocity

$$\vec{v} = l\dot{\Theta}\,\hat{\Theta} + \omega l \sin\Theta\,\hat{\phi}$$

$$T = 2\left(\tfrac{1}{2}M\vec{v}\cdot\vec{v}\right) = M\left(l^2\dot{\Theta}^2 + \omega^2 l^2 \sin^2\Theta\right)$$

$$L = T - V = Ml^2\dot{\theta}^2 + M\omega^2 l^2 \sin^2\theta + 2Mgl\cos\theta$$

$$\frac{\partial L}{\partial \theta} = 2M\omega^2 l^2 \sin\theta\cos\theta - 2Mgl\sin\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 2Ml^2\ddot{\theta}$$

<u>EOM</u>

$$0 = \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 2M\omega^2 l^2 \sin\theta\cos\theta - 2Mgl\cos\theta - 2Ml^2\ddot{\theta} = 0$$

$$\boxed{\ddot{\theta} - \omega^2 \sin\theta\cos\theta + g/l\,\sin\theta = 0}$$

$$\boxed{\cos\theta_0 = \frac{g}{l\omega^2}}$$ At equilibrium $\ddot{\theta} = 0$

Let $\cancel{\theta' =}$ $\theta = \theta' + \theta_0$, θ' small

$$\ddot{\theta} = \ddot{\theta}'$$

$$\boxed{\sin 2\theta = 2\cos\theta\sin\theta}$$

$$\ddot{\theta} - \frac{\omega^2}{2}\sin 2\theta + g/l\,\sin\theta = 0$$

$$\ddot{\theta}' - \frac{\omega^2}{2}\sin(2\theta' + 2\theta_0) + \frac{g}{l}\sin(\theta' + \theta_0) = 0$$

Expand $\sin 2\theta$, $\sin\theta$ about θ_0 to first order.

$$\sin\theta \sim \left.\frac{d\sin\theta}{d\theta}\right|_{\theta_0}\theta' = \cos\theta_0\,\theta'$$

$$\sin 2\theta \sim \left.\frac{d\sin 2\theta}{d\theta}\right|_{\theta_0}\theta' = 2\cos 2\theta_0\,\theta'$$

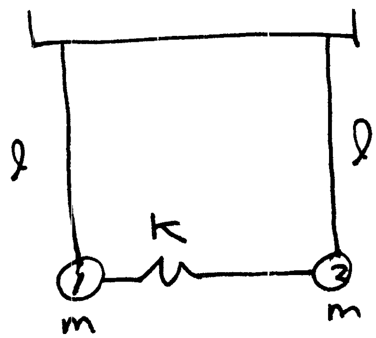
Discarding constant term which is zero because of equilibrium.

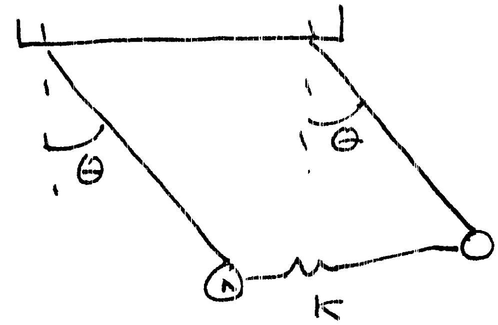
Linearized EOM

$$\ddot{\theta}' - \omega^2\cos 2\theta_0\,\theta' + \frac{g}{9}\cos\theta_0\,\theta' = 0$$

$$\ddot{\theta}' + \underbrace{\left(\frac{g}{9}\cos\theta_0 - \omega^2\cos 2\theta_0\right)}_{\omega_s^2}\theta' = 0$$

Double Pendula





$$V = mgl(1-\cos\theta_1) + mgl(1-\cos\theta_2) + \cancel{\tfrac{1}{2}k\left[l(\sin\theta_1 - \sin\theta_2)\right]^2}$$
$$+ \frac{1}{2}kl^2(\sin\theta_1 - \sin\theta_2)^2$$

Expand to $O(\theta^2)$

$$V = \cancel{mgl} + \frac{1}{2}mgl\theta_1^2 + \cancel{mgl} + \frac{1}{2}mgl\theta_2^2$$
$$+ \frac{1}{2}kl^2(\theta_1 - \theta_2)^2$$

$$\sin\theta \sim \theta \qquad \cos\theta \sim 1 - \frac{1}{2}\theta^2$$

$$T = \frac{1}{2}ml^2\dot{\theta}_1^2 + \frac{1}{2}ml^2\dot{\theta}_2^2$$

$$L = T - V = \frac{1}{2} m l^2 \dot{\theta}_1^2 + \frac{1}{2} m l^2 \dot{\theta}_2^2$$
$$- \frac{1}{2} mgl\theta_1^2 - \frac{1}{2} mgl\theta_2^2$$
$$- \frac{1}{2} kl^2 (\theta_1 - \theta_2)^2$$

<u>EOM θ_1</u>

$$\frac{\partial L}{\partial \theta_1} = -mgl\theta_1 - kl^2(\theta_1 - \theta_2)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} = ml^2\ddot{\theta}_1$$

$$0 = \frac{\partial L}{\partial \theta_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} = -mgl\theta_1 - kl^2\theta_1 + kl^2\theta_2 - ml^2\ddot{\theta}_1 = 0$$

$$\ddot{\theta}_1 + \left(\frac{g}{l} + \frac{k}{m}\right)\theta_1 - \frac{k}{m}\theta_2 = 0$$

<u>Define</u> $\omega_p^2 = \frac{g}{l}$ $\quad \omega_s^2 = \frac{k}{m}$

$$\ddot{\theta}_1 + (\omega_p^2 + \omega_s^2)\theta_1 - \omega_s^2\theta_2 = 0$$

EOM θ_2

$$\frac{\partial L}{\partial \theta_2} = -mgl\theta_2 + kl^2(\theta_1 - \theta_2)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta_2} = ml^2\ddot\theta_2$$

$$0 = \frac{\partial L}{\partial \theta_2} - \frac{d}{dt}\frac{\partial L}{\partial \dot\theta_2} = -mgl\theta_2 - kl^2\theta_2 + kl^2\theta_1 - ml^2\ddot\theta_2$$

$$\ddot\theta_2 + (\omega_p^2 + \omega_s^2)\theta_2 - \omega_s^2\theta_1 = 0$$

Propose solution

$$\theta_1 = A_1\cos(\omega t + \sigma) \qquad \theta_2 = A_2\cos(\omega t + \sigma)$$

$$\ddot\theta_1 = -\omega^2 A_1\cos(\omega t + \sigma) \qquad \ddot\theta_2 = -\omega^2 A_2\cos(\omega t + \sigma)$$

Substitute into EOM

θ_1 EOM

$$-\omega^2 A_1 + (\omega_p^2 + \omega_s^2)A_1 - \omega_s^2 A_2 = 0$$

θ_2 EOM

$$-\omega^2 A_2 + (\omega_p^2 + \omega_s^2) A_2 - \omega_s^2 A_1 = 0$$

Matrix

$$\begin{pmatrix} -\omega^2 + \omega_p^2 + \omega_s^2 & -\omega_s^2 \\ -\omega_s^2 & -\omega^2 + \omega_p^2 + \omega_s^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

$$\det(\;) = 0 = (\omega_p^2 + \omega_s^2 - \omega^2)^2 - \omega_s^4 = 0$$

$$(\omega_p^2 + \omega_s^2)^2 - 2\omega_p^2\omega^2 - 2\omega_s^2\omega^2 + \omega^4 - \omega_s^4 = 0$$

$$\omega^4 - 2(\omega_p^2 + \omega_s^2)\omega^2 + \omega_p^4 + 2\omega_p^2\omega_s^2 + \omega_s^4 - \omega_s^4 = 0$$

$$\omega^4 - 2(\omega_p^2 + \omega_s^2)\omega^2 + \omega_p^4 + 2\omega_p^2\omega_s^2 = 0$$

$$\omega^2 = \frac{2(\omega_p^2+\omega_s^2) \pm \sqrt{4(\omega_p^2+\omega_s^2)^2 - 4(\omega_p^4+2\omega_p^2\omega_s^2)}}{2}$$

$$= \omega_p^2+\omega_s^2 \pm \omega_s^2$$

<u>Normal Frequencies</u>

$$\omega_1^2 = \omega_p^2 \qquad \omega_2^2 = \omega_p^2 + 2\omega_s^2$$

<u>Normal Coordinates</u> $\omega_1^2 = \omega_p^2$

$$\begin{pmatrix} -\omega_p^2 + \omega_p^2 + \omega_s^2 & -\omega_s^2 \\ -\omega_s^2 & -\omega_p^2 + \omega_p^2 + \omega_s^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} \omega_s^2 & -\omega_s^2 \\ -\omega_s^2 & \omega_s^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

$$A_1\omega_s^2 - A_2\omega_s^2 = 0$$

If $A_1 = 1$, $A_2 = 1$

$$Q_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Normal Coordinates $\omega_2^2 = \omega_p^2 + 2\omega_s^2$

$$\begin{pmatrix} -(\omega_p^2+2\omega_s^2)+\omega_p^2+\omega_s^2 & -\omega_s^2 \\ -\omega_s^2 & -(\omega_p^2+2\omega_s^2)+\omega_p^2+\omega_s^2 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} -\omega_s^2 & -\omega_s^2 \\ -\omega_s^2 & -\omega_s^2 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

If $A_1 = 1$, $A_2 = -1$

$$Q_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

General Solution

$$\cancel{\theta_1(t)} = \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} = A\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos(\omega_1 t+\delta_1) + B\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos(\omega_2 t+\delta_2)$$

Suppose the left mass is struck and given an initial velocity v_0 while $\theta_1 = 0$, $\theta_2 = 0$, $\dot{\theta}_2 = 0$.

Initial condition

$$\begin{pmatrix} \theta_1(0) \\ \theta_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = A \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos \delta_1 + B \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos \delta_2$$

$$A \cos \delta_1 + B \cos \delta_2 = 0$$

$$A \cos \delta_1 - B \cos \delta_2 = 0$$

$$2A \cos \delta_1 = 0$$

$$2B \cos \delta_2 = 0$$

Try $\delta_1 = \delta_2 = \pi/2$

$$\begin{pmatrix} \dot{\theta}_1(0) \\ \dot{\theta}_2(0) \end{pmatrix} = \begin{pmatrix} v_0/l \\ 0 \end{pmatrix} = -A\omega_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin \delta_1 - B\omega_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin \delta_2$$

$$\frac{v_0}{l} = -A\omega_1 \sin \delta_1 - B\omega_2 \sin \delta_2 = -A\omega_1 - B\omega_2$$

$$0 = -A\omega_1 \sin \delta_1 + B\omega_2 \sin \delta_2 = -A\omega_1 + B\omega_2$$

$$\frac{v_0}{l} = -2A\omega_1 \qquad A = -\frac{v_0}{\omega_1 l}$$

$$B = A\frac{\omega_1}{\omega_2} \qquad B = -\frac{v_0}{\omega_2 l}$$

Solution

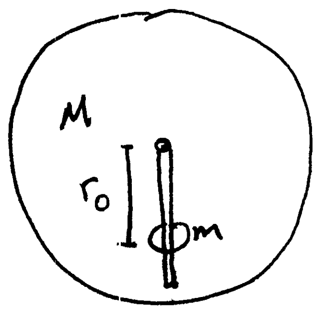
$$\begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} = \frac{-V_0}{\omega_1 \ell} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos\left(\omega_1 t + \frac{\pi}{2}\right)$$

$$- \frac{V_0}{\omega_2 \ell} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos\left(\omega_2 t + \frac{\pi}{2}\right)$$

Ex Disk of mass M and radius a is restored to equilibrium by a torsion spring $V = \frac{1}{2} k_T a^2 (\theta - \theta_0)^2$.

A bead of mass m is constrained to move on a rod fixed to the disk. The bead has a restoring force with potential $V = \frac{1}{2} k_s (r - r_0)^2$

At equilibrium the bead hangs straight down.



$I_{disk} = \frac{1}{2} M a^2$

The system is configured so $\theta_0 = 0$. r_0 is the unstretched length of the spring.

Let θ be the angle from ~~vertical~~ vertical.

$$T = \underbrace{\frac{1}{2} I \dot{\theta}^2}_{T_{disk}} + \underbrace{\frac{1}{2} m \left(\dot{r}^2 + (r\dot{\theta})^2\right)}_{T_m = \frac{1}{2} m \vec{v} \cdot \vec{v}}$$

$$V = \frac{1}{2} k_T a^2 (\theta)^2 + \frac{1}{2} k_s (r - r_0)^2 \underbrace{- mgr\cos\theta}$$

zero of potential at center.

Find Equilibrium

$$\frac{\partial V}{\partial \theta} = k_T a^2 \theta + mgr\sin\theta = 0 \qquad \theta_0 = 0$$

$$\frac{\partial V}{\partial r} = k_s (r - r_0) - mg\cos\theta = 0$$

$$k_s r_{eq} = mg + k_s r_0 \qquad \text{if } \theta_0 = 0$$

$$k_{11} = \frac{\partial^2 V}{\partial \theta^2} = k_T a^2 + mgr\cos\theta \Big|_{r_{eq}, \theta_0} = k_T a^2 + mgr_{eq}$$

$$k_{22} = \frac{\partial^2 V}{\partial r^2} = k_s$$

$$\frac{\partial^2 V}{\partial r \partial \theta} = mg\sin\theta \Big|_{\theta_0=0} = 0$$

$$V = V_0 + \frac{1}{2}\left(k_{11}(\theta-\theta_0)^2 + k_{22}(r-r_{eq})^2 \right)$$

Let $r' = r - r_{eq}$ $\qquad \theta' = \theta - \theta_0 = 0$

$$T = \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} m \dot{r}'^2 + \frac{1}{2} m (r' + r_{eq})^2 \dot{\theta}^2$$

Keep terms to second order

$$T = \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} m \dot{r}'^2 + \frac{1}{2} m r_{eq}^2 \dot{\theta}^2$$

$$L = T - V = \frac{1}{2}\left(I + m r_{eq}^2\right)\dot{\theta}^2 + \frac{1}{2} m \dot{r}'^2$$

$$- \frac{1}{2} k_{11} \theta^2 - \frac{1}{2} k_{22} r'^2$$

System not coupled, we can read off normal mode frequencies.

$$\omega_\theta^2 = \frac{k_{11}}{I + m r_{eq}^2} \qquad \omega_r^2 = \frac{k_{22}}{m} = \frac{k_s}{m}$$