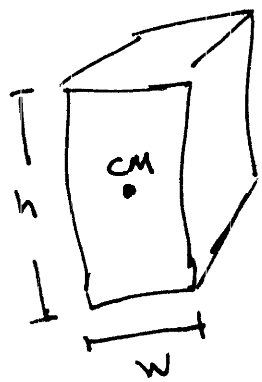
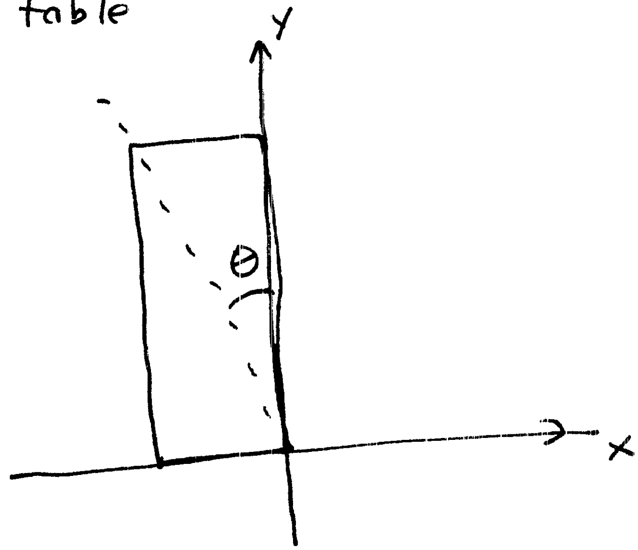
# Stability

Consider the potential energy curve of a brick of mass $m$ sitting on a table

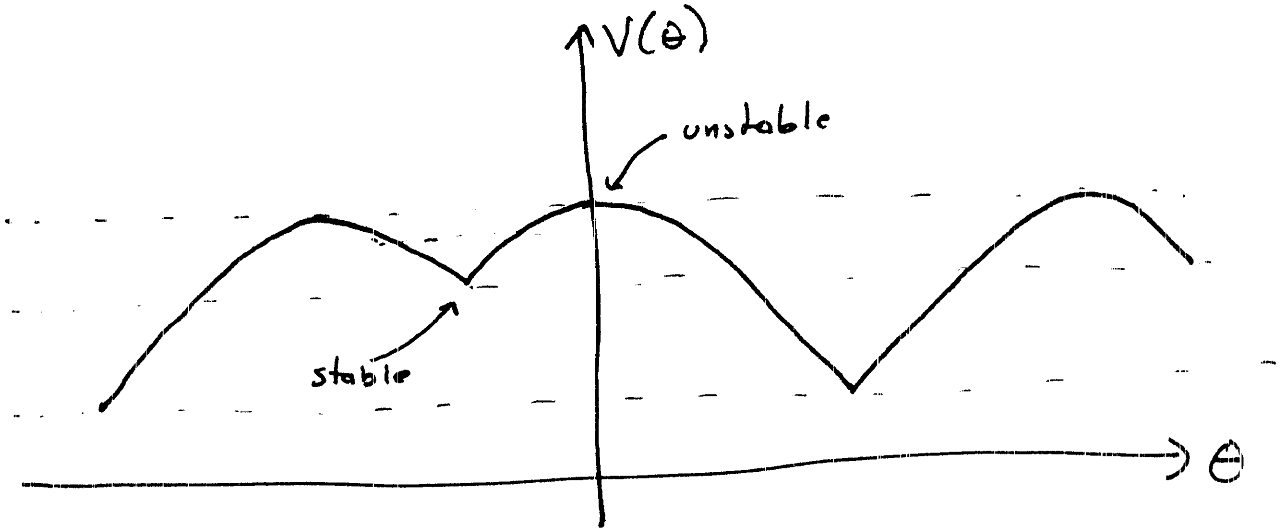



$\tan(\theta_0) = -\frac{w}{h}$ $\qquad \theta_0$ = angle when sitting flat on table

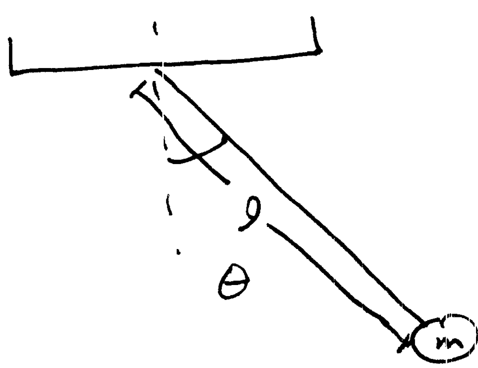
$$V = mg\frac{h}{2}\cos\theta$$

Simply the height of the center of mass.




Minima of $V(\theta)$ are points of stable equilibrium, since the force $Q_\theta = -\frac{\partial V}{\partial \theta}$ points to lower potential and therefore push the system back to equilibrium if displaced.

Another more familiar system - Simple Pendulum




$$V(\theta) = mgl(1 - \cos\theta)$$

Points of equilibrium satisfy $Q_\theta = 0$

$$\Rightarrow \frac{dV}{d\theta} = 0$$

$$\frac{dV}{d\theta} = mgl \sin\theta_0 = 0 \qquad \theta_0 = 0, \pi, \ldots$$

Points of stable equilibrium satisfy

$$\left.\frac{d^2V}{d\theta^2}\right|_{\theta_0} > 0 \qquad \frac{d^2V}{d\theta^2} = mgl\cos\theta$$

so $\theta_0 = 0$ is a point of stable equilibrium

and $\theta_0 = \pi$ is unstable.

---

Define $V_0'' = \left.\frac{d^2V}{d\theta^2}\right|_{\theta_0}$

We can expand about equilibrium point and investigate small oscillations about equilibrium.

$$V(\theta) \sim V(\theta_0) + \frac{1}{2} V_0''(\theta - \theta_0)^2 + \cdots$$

Change variables so equilibrium location is zero,

$$\theta' = \theta - \theta_0$$

$$V(\theta') \sim V(0) + \frac{1}{2} V_0'' \theta'^2$$

Lagrangian

$$L = \frac{1}{2} I \dot{\theta}^2 - \frac{1}{2} V_0'' \theta^2$$

Constant terms can be discarded because they do not affect the dynamics.

EOM

$$-V_0''\theta - I\ddot{\theta} = 0$$

$$\ddot{\theta} + \omega_0^2 \theta = 0 \qquad \omega_0^2 = \frac{V_0''}{I} = \frac{mgl}{ml^2} = g/l$$

If ~~sp~~ position coordinate

$$-V_0'' q - m\ddot{q} = 0 \qquad \omega_0^2 = \frac{V_0''}{m} \qquad V_0'' = K$$

Consider $q_n$ degrees of freedom

$$V(q_1, q_2 \cdots q_n)$$

Points of equilibrium satisfy

$$\frac{\partial V}{\partial q_1} = \frac{\partial V}{\partial q_2} \cdots = \frac{\partial V}{\partial q_n} = 0$$

We can expand about a point of equilibrium

$$\vec{q}_0 = (q_1^0, q_2^0 \cdots q_n^0)$$

$$V = V(\vec{q}_0) + \frac{1}{2}\Big( K_{11}(q_1 - q_1^0)^2 + 2K_{12}(q_2 - q_2^0)(q_1 - q_1^0) + K_{22}(q_2 - q_2^0)^2 \cdots \Big)$$

where

$$K_{ij} = \left.\frac{\partial^2 V}{\partial q_i \partial q_j}\right|_{\vec{q}_0}$$

The equilibrium is stable if the stuff in ( ) is positive or zero. This will occur if the principle minors from the left corner (Mathworld) are positive

$$K_{11} > 0 \quad \text{and} \quad \begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} > 0 \quad \text{and}$$

$$\begin{vmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{vmatrix} > 0$$

# Coupled Oscillations

Now consider small oscillations about a stable equilibrium point.

$$V = \frac{1}{2} \sum K_{ij} q_i q_j$$

where we have changed coordinates so the equilibrium position is zero, and redefined the zero of potential.

This can be found by expanding the potential function and discarding terms $O(q^3)$.

Ex $\quad V = -\cos(x/a)\cos(y/b)$

$$\cos x = 1 - \frac{1}{2}x^2 + \ldots \; O(x^3)$$

$$V \cong -\left(1 - \frac{1}{2}\left(\frac{x}{a}\right)^2\right)\left(1 - \frac{1}{2}\left(\frac{y}{b}\right)^2\right)$$

$$= -1 + \frac{1}{2}\left(\frac{x}{a}\right)^2 + \frac{1}{2}\left(\frac{y^2}{b}\right) + O(xy^3)$$

$$V = \frac{1}{2}K_{11}x^2 + \frac{1}{2}K_{22}y^2$$

$$K_{11} = \frac{1}{a^2} \qquad K_{22} = \frac{1}{b^2}$$

$q_1 = x$

$q_2 = y$

Equilibrium stable because

$$K_{11} > 0 \qquad \begin{vmatrix} K_{11} & 0 \\ 0 & K_{22} \end{vmatrix} = K_{11} K_{22} > 0$$

Consider a general kinetic energy term

$$T = \frac{1}{2} \sum M_{ij} \dot{q}_i \dot{q}_j$$

also only $O(q^2)$.

For example, if mass $m$ moved in the potential above

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2$$

Let $q_1 = x \quad q_2 = y$

$$M_{11} = m \qquad M_{22} = m$$
$$M_{12} = M_{21} = 0$$

$$T = \sum_{1}^{2} M_{ij} \dot{q}_i \dot{q}_j$$

## Lagrangian

$$L = \frac{1}{2}\sum M_{ij}\,\dot{q}_i\,\dot{q}_j \;-\; \frac{1}{2}\sum_{ij} K_{ij}\,q_i\,q_j$$

## EOM $q_k$

$$0 = \frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = 0$$

$$-\sum_i \overset{K_{ki}}{\cancel{K_{kk}}}\,q_i \;-\; \sum_i \overset{M_{ki}}{\cancel{M_{ik}}}\,\ddot{q}_{\cancel{k}\,i} = 0$$

$$\sum_i \left(K_{ki}\,q_i + M_{ki}\,\ddot{q}_i\right) = 0$$

Let $\vec{q}$ be the vector $\begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$

$$\overleftrightarrow{K} = \begin{pmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ \vdots & & & \\ K_{n1} & \cdots & \cdots & K_{nn} \end{pmatrix} \qquad \overleftrightarrow{M} = \begin{pmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & & \\ M_{n1} & \cdots & M_{nn} \end{pmatrix}$$

Our n EOM become the matrix multiplication

$$\overleftrightarrow{M}\ddot{\vec{q}} + \overleftrightarrow{K}\vec{q} = 0$$

The example earlier becomes

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = 0$$

## Normal Modes

For certain specific exitations, the coupled system will oscillate with a single frequency, a normal (or collective) mode.

$\Longrightarrow$ For most excitation, many frequencies of oscillation are involved.

$\Longrightarrow$ One normal mode for each degree of freedom

Propose a solution where the whole system oscillates with the same frequency, a collective motion.

$$q_i = A_i \cos(\omega t - \sigma)$$

$$\ddot{q}_i = -\omega^2 A_i \cos(\omega t - \sigma)$$

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} -\omega^2 A_1 \cos(\omega t - \sigma) \\ -\omega^2 A_2 \cos(\omega t - \sigma) \end{pmatrix} + \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{pmatrix} \begin{pmatrix} A_1 \cos(\omega t - \sigma) \\ A_2 \cos(\omega t - \sigma) \end{pmatrix} = 0$$

$$\begin{pmatrix} \frac{1}{a^2} - m\omega^2 & 0 \\ 0 & \frac{1}{b^2} - m\omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \cos(\omega t - \sigma) = 0$$

This linear system has a solution iff $\det(\;) = 0$

$$\det \begin{pmatrix} \frac{1}{a^2} - m\omega^2 & 0 \\ 0 & \frac{1}{b^2} - m\omega^2 \end{pmatrix} = \left(\frac{1}{a^2} - m\omega^2\right)\left(\frac{1}{b^2} - m\omega^2\right)$$

$$= 0$$

## Normal frequencies

$$\omega_1^2 = \frac{1}{ma^2} \qquad \omega_2^2 = \frac{1}{mb^2}$$

To see a pure $\omega_1$ oscillation, we have to select $A_1$, $A_2$ properly. ~~There~~ The vector $\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ is the eigenvector of $\left(\quad\right)$.

Find eigenvector $\omega_1^2$

$$\begin{pmatrix} \frac{1}{a^2} - m\left(\frac{1}{ma^2}\right) & 0 \\ 0 & \frac{1}{b^2} - m\left(\frac{1}{ma^2}\right) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{b^2} - \frac{1}{a^2} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

$$\Rightarrow \quad A_2 = 0$$

Eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

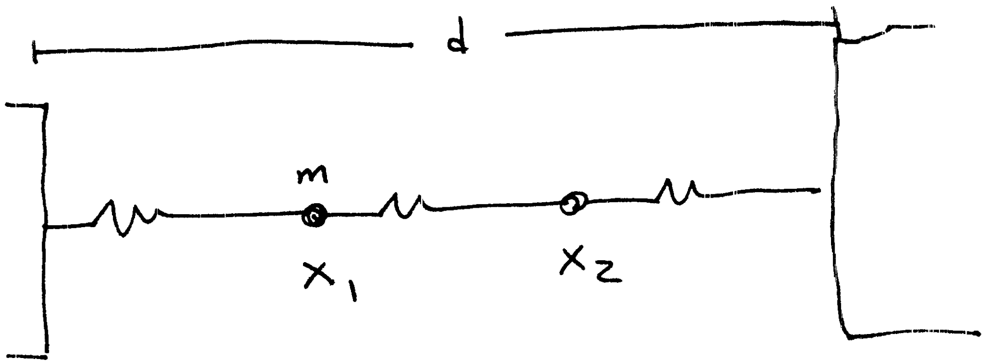
Likewise, the eigenvector for $\omega_2^2$ is

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So to excite a pure $\omega_1$ oscillation we displace the mass in the x direction but not the y direction which is fairly obvious.

- The normal modes and their eigenvectors define a new coordinate system where all oscillators are decoupled.
- Just as in QM this is accomplished by using the eigenvectors to the ( ) until it is diagonal.
- This means the $\omega^2 = 0$ modes rotate, & the equation $\ddot{q} + \omega^2 q = \ddot{q}$. The solution to this is $q(t) = At + B$ not $A\cos(\omega t + \delta)$

Three equal mass on springs with spring constant $k$ between two supports a distance $d$ apart. The equilibrium length of the springs are $x_0$




Find equilibrium location of masses

$$V(x_1, x_2) = \frac{1}{2}k(x_1 - x_0)^2 + \frac{1}{2}k(d - x_2 - x_0)^2$$
$$+ \frac{1}{2}k(x_2 - x_1 - x_0)^2$$

<u>Equilibrium</u>

$$\frac{\partial V}{\partial x_1} = k(x_1 - x_0) - k(x_2 - x_1 - x_0) = 0$$

$$= 2kx_1 - kx_2 = 0$$

$$x_2 = 2x_1$$

$$\frac{\partial V}{\partial x_2} = -k(d - x_2 - x_0) + k(x_2 - x_1 - x_0) = 0$$

$$2x_2 - x_1 = d$$

$$4x_1 - x_1 = d$$

$$x_1^0 = d/3 \qquad x_2^0 = 2d/3$$

Equilibrium

Let $\quad x_1' = x_1 - x_1^0 \qquad\qquad x_2' = x_2 - x_2^0$

Expand $V$ about equilibrium

$$K_{11} = \left.\frac{\partial^2 V}{\partial x_1^2}\right|_{x^0} = 2k$$

$$K_{22} = \left.\frac{\partial^2 V}{\partial x_2^2}\right|_{x_0} = 2k$$

$$K_{12} = \left.\frac{\partial^2 V}{\partial x_1 \partial x_2}\right|_{x_0} = -k$$

$$V = \frac{1}{2}\left( k_{11} x_1'^2 + k_{22} x_2'^2 + 2 k_{12} x_1' x_2' \right)$$

$$= \frac{1}{2}\left( 2k x_1'^2 + 2k x_2'^2 - 2k x_1' x_2' \right)$$

$$= \frac{1}{2}\left( k x_1'^2 + k x_2'^2 + k(x_2' - x_1')^2 \right)$$

what we would have written if we assumed the masses were in equilibrium and worked with the displacements.

---

Drop primes

$$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$$

$$L = T - V = \frac{1}{2} m\left(\dot{x}_1^2 + \dot{x}_2^2\right) - \frac{1}{2} k\left(2x_1^2 + 2x_2^2 - 2x_1 x_2\right)$$

$x_1$ EOM

$$\frac{\partial L}{\partial x_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} = -2k x_1 + k x_2 - m\ddot{x}_1 = 0$$

$x_2$ EOM

$$\frac{\partial L}{\partial x_2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} = -2k x_2 + k x_1 - m\ddot{x}_2 = 0$$

## Propose Normal Mode

$$x_1 = A_1 \cos(\omega t + \delta)$$

$$x_2 = A_2 \cos(\omega t + \delta)$$

### $x_1$ EOM

$$-2k A_1 + k A_2 + m\omega^2 A_1 = 0$$

$$\omega_0^2 \equiv \frac{k}{m}$$

$$-2\omega_0^2 A_1 + \omega_0^2 A_2 + \omega^2 A_1 = 0$$

### $x_2$ EOM

$$-2k A_2 + k A_1 + \omega^2 m A_2 = 0$$

$$-2\omega_0^2 A_2 + \omega_0^2 A_1 + \omega^2 A_2 = 0$$

### Write as matrix

$$\begin{pmatrix} \omega^2 - 2\omega_0^2 & \omega_0^2 \\ \omega_0^2 & \omega^2 - 2\omega_0^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

Solution exists if $\det(\,) = 0$

$$(\omega^2 - 2\omega_0^2)^2 - \omega_0^4 = 0$$

$$\omega^4 - 4\omega^2\omega_0^2 + 3\omega_0^4 = 0$$

$$(\omega^2 - 3\omega_0^2)(\omega^2 - \omega_0^2) = 0$$

Normal modes $\omega_1^2 = \omega_0^2$ $\quad \omega_2^2 = 3\omega_0^2$

---

Now find the normal coordinates, the eigenvectors of the eigenvalues $\omega_i$. These separate the oscillation into orthogonal modes each oscillating with $\omega_i$.

The normal ~~mod~~ coordinate vector

$$Q_i = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

shows how the system must be prepared to oscillate only with frequency $\omega_i$.

Eigenvector $\omega_1^2 = \omega_0^2$

$$\begin{pmatrix} \omega_0^2 - 2\omega_0^2 & \omega_0^2 \\ \omega_0^2 & \omega_0^2 - 2\omega_0^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

$$-\omega_0^2 A_1 + \omega_0^2 A_2 = 0$$

If $A_1 = 1$, $A_2 = 1$

$$Q_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Symmetric Mode - Displace both masses in $+x$ direction

Eigenvector $\omega_2^2 = 3\omega_0^2$

$$\begin{pmatrix} 3\omega_0^2 - 2\omega_0^2 & \omega_0^2 \\ \omega_0^2 & 3\omega_0^2 - 2\omega_0^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

$$\omega_0^2 A_1 + \omega_0^2 A_2 = 0$$

If $A_1 = 1$, $A_2 = -1$

$$Q_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Anti-symmetric mode, masses displaced in opposite directions

The general solution can be written

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos \omega_1 t + \delta_1 + A_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_2 t + \delta_2)$$

To excite a pure $\omega_1$ oscillation use initial condition $x_1 = A$ $x_2 = A$ $\delta_1 = 0$

$$\cancel{x_1 = A_1 \quad x_2 = A_2 \quad \delta_1 = 0}$$