Example Test 2

Justin' s version

Modified From Question 1.19

A bee goes out from its hive in a spiral path given in plane polar coordinates by
$r = b e^{kt}$, θ=ct
where b k and c are positive (real) constants.
1) Show that the angle between the velocity vector and the acceleration vector remains constant as the bee movees outward. (Hint: Find **v**·**a**/va)
2) What is the force on the bee in plane polar coordinates?

Solution

1)

Find velocity
$\mathbf{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$
$\mathbf{a} = \left(\ddot{r} - r\,\dot{\theta}^2\right)\hat{r} + \left(r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta}\right)\hat{\theta}$

$\mathrm{v} = \mathrm{kb}e^{\mathrm{kt}}\,\hat{r} + \mathrm{cb}e^{\mathrm{kt}}\,\hat{\theta}$

$\mathrm{a} = \left(\mathrm{k}^2\,\mathrm{b}e^{\mathrm{kt}} - \mathrm{b}e^{\mathrm{kt}}\,\mathrm{c}^2\right)\,\hat{r} + \left(-2\,\mathrm{ckb}e^{\mathrm{kt}}\right)\,\hat{\theta}$
$= \mathrm{b}e^{\mathrm{kt}}\,\left(\mathrm{k}^2 - \mathrm{c}^2\right)\,\hat{r} - 2\,\mathrm{ck}e^{\mathrm{kt}}\,\hat{\theta}$

Find dot product. If this is constant then the angle between the two vectors is also constant.
Remember how to to derivatives in polar coordinates.

$\mathbf{v}\cdot\mathbf{a} = \mathrm{kb}^2\left(k^2 - c^2\right)e^{2\,\mathrm{kt}} - 2\,\mathrm{kc}^2\,e^{2\,\mathrm{kt}}$
$\quad = e^{2\,\mathrm{kt}}\left(\mathrm{kb}^2\left(k^2 - c^2\right) - 2\,\mathrm{kc}^2\right)$

$\mathrm{va} = \left(k^2\,b^2\,e^{2\,\mathrm{kt}} + c^2\,b^2\,e^{2\,\mathrm{kt}}\right)\left(b^2\left(k^2 - c^2\right)^2 e^{2\,\mathrm{kt}} + 4\,c^2\,k^2\,e^{2\,\mathrm{kt}}\right)$
$\quad = e^{2\,\mathrm{kt}}\left(k^2\,b^2 + c^2\,b^2\right)\left(b^2\left(k^2 - c^2\right)^2 + 4\,c^2\,k^2\right)$

$$\frac{\boldsymbol{v}\cdot\boldsymbol{a}}{\mathrm{va}} = \frac{\left(\mathrm{kb}^2\left(k^2-c^2\right)-2\,\mathrm{kc}^2\right)}{\left(k^2\,b^2+c^2\,b^2\right)\left(b^2\left(k^2-c^2\right)^2+4\,c^2\,k^2\right)}$$

The above has no time depnednce. It looks terrible, but who cares? No time dependence is a good thing.

2)

Force is still F=ma.

$$F(t) = m\, e^{k\,t}\left(b\left(k^2 - c^2\right) \hat{r} \;-\; 2\,k\,c^2\, e^{k\,t}\, \hat{\theta}\right)$$

Modified from old HW 2 E1 (perhaps yours as well)

A system with two stable states can be represented by a potentia in the form U(x)=a - bx^2 + cx^4
1) Find the location of the minima of this potential
2) What condition must be met for the two minima to exist?
3) When the potential has two minima, find the height of the energy barrier a particle at the bottom of one of the minima must overcome to move to the other stable minima.
4) When the particle has two minima, how fast would the particle released from the origin be moving when it reached the positive minima?

Solution

1)

```
U[x_] = a - b x^2 + c x^4
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$a - b x^2 + c x^4$

take first derivative of U[x]

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slope[x_] = ∂x U[x]
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$-2 b x + 4 c x^3$

Solve for where the derivative is equal to zero

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Solve[slope[x] == 0, x]
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$\left\{\{x \to 0\}, \left\{x \to -\frac{\sqrt{b}}{\sqrt{2}\,\sqrt{c}}\right\}, \left\{x \to \frac{\sqrt{b}}{\sqrt{2}\,\sqrt{c}}\right\}\right\}$

Look at the values of these points.

$\mathbf{U}\left[\left\{0, -\frac{\sqrt{b}}{\sqrt{2}\,\sqrt{c}}, \frac{\sqrt{b}}{\sqrt{2}\,\sqrt{c}}\right\}\right]$

$\left\{a, a - \frac{b^2}{4 c}, a - \frac{b^2}{4 c}\right\}$

Two are identical (we know this should be symmetric, so we' ll go with these.
U($\frac{\sqrt{b}}{\sqrt{2}\,\sqrt{c}}$) = $a - \frac{b^2}{4 c}$ is a minimum.

2)

Look at the second derivatives at these likely minima.

Compute second derivative

```
∂x,x U[x]
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$-2 b + 12 c x^2$

Plug in for out desired values.

```
curve[x_] = ∂x,x U[x]
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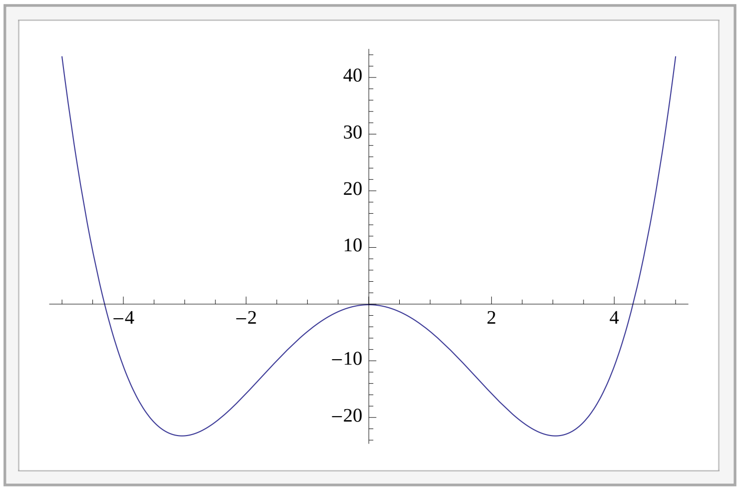
$-2 b + 12 c x^2$

$$\text{curve}\left[\left\{-\frac{\sqrt{b}}{\sqrt{2}\,\sqrt{c}},\ \frac{\sqrt{b}}{\sqrt{2}\,\sqrt{c}}\right\}\right]$$

{4 b, 4 b}

We want the constant b to be to be positive so that the second derivative shows a stable point.
It will look like the following plot.

```
Manipulate[Plot[a - b x^2 + c x^4, {x, -5, 5}], {a, -1, 1}, {b, -5, 5}, {c, -3, 3}]
```



3)

Energy barrier is equal to the value of the energy at x = 0. You know this because x=0 is the only possible value for a local maximum.

```
U[0]
```

a

4)

Conserve energy to get the final velocity.
Find potential energy of the minimum.

$$U\left[\frac{\sqrt{b}}{\sqrt{2}\,\sqrt{c}}\right]$$

$$a - \frac{b^2}{4 c}$$

$U(0) = a, \quad U\left(\frac{\sqrt{b}}{\sqrt{2}\,\sqrt{c}}\right) = a - \frac{b^2}{4c}$

$T(\text{minima}) = -\Delta\text{U}$

$\Delta\text{U} = a - \frac{b^2}{4c} - a = -\frac{b^2}{4c}$

$T = \frac{1}{2} m v^2 = \frac{b^2}{4c}$

$v = b\sqrt{\frac{1}{2\,c\,m}}$

Modified From Question 3.11

A mass m moves along the x-axis subject to an attractive force given by $17\beta^2\, mx/2$ and a retarding force given by $3\beta m\dot{x}$, where x is the distance from the origin and β is a constant. A driving force given by mAcos(ωt), where A is a constant, is applied to the particle along the x-axis.
a) Write the equation of motion for this system.
b) What value of ω results in a stedy-state oscillations about the origin with maximum amplitude?
c) What is the maximum amplitude?

Solution

Tunnel problem

You dig a tunnel through the Earth. Let's just assume your some kind of crazy disaster movie dilling expert. The tunnel goes though the center and keeps going to the opposite side of the plannet. You drop a ball of mass m down this tunnel. Let M_e be the Earth's mass and R_e be it's radius.

1) How fast is the ball traveling when it reaches the center of the Earth? Write your answer in terms of m, M and R_e. Use the fact that the force on the ball can be written as the following.

$F = G \frac{\rho m r}{3}$ where

$\rho = \frac{3 M_e}{4 \pi R_e^3}$.

2) How long does it take for the ball to get to the center of the Earth?

Solution

Nathan's Gauss Gun Near a Fan

Nathan build a seriously cool Gauss Gun when he was in UP II. It worked well and was actually portable. Assume that Nathan fires it horizontally ($\hat{x}$) when under a ceiling fan. The fan is pointed down $(-\hat{z})$.

1) Use the following information to construct the remaining equations of motion. Let c be the linear drag coefficient and let v_{air} be a constant.

$\ddot{z} = -g - c\,(v_{\text{air}} + \dot{z})$

$\vec{v}_{\text{air}} = -v_{\text{air}}\,\hat{z}$

$\vec{v}_{\text{ball}}(t=0) = v_0\,\hat{x}$

2) Solve the equations of motion to write the trajectory of the steel ball Nathan just shot.

3) Now assume we turn off the ceiling fan and turn on a fan facing the $\hat{y}$ direction pushing air at $\vec{v} = v_{\text{air}}\,\hat{y}$. Write the equations of motion for this case. Don't worry about solving them.

Solution

1)

We already know the $\ddot{z}$ equation.

$\ddot{z} = -(g + c\,v_{\text{air}}) - c\,\dot{z}$

The x and y parts are much simpler. Just as they would be from the reading.

$\ddot{x} = -c\,\dot{x}$

$\ddot{y} = -c\,\dot{y}$

2)

Solving is easier than you may think. The x and y parts are just what they would have been in exmaples from the book.

$\dot{x} = C_1 e^{-c\,t} + C_2$
where from our initial conditions $C_1 = v_0$ and $C_2 = 0$.
Integrate to get x(t).
$x(t) = \frac{v_0}{c}(1 - e^{-c\,t}) + x_0$

y(t) would work in exaclty the same way, except $\dot{y}(t = 0) = 0$. Thus,

$y(t) = y_0$

$z(t)$ is slightly more complicated since it is **non**-homogenious. Solveing this type of differential requires a homogenious and particular solution.

$\dot{z}_{\text{hom}} = C_3 e^{-c\,t}$, $\dot{z}_{\text{part}} = \frac{g + c\,v_{\text{air}}}{c}$

Add these together to get $\dot{z}(t)$.

$\dot{z}(t) = C_3 e^{-c\,t} + \frac{g + c\,v_{\text{air}}}{c}$
Differentiate to prove this gives the corret result.

Integrate to get $z(t)$.
$z\,(t) = \frac{C_3}{c} e^{-c\,t} - \frac{g + c\,v_{\text{air}}}{c}\,t + C_4$

We can use out initial conditions to find C_3 and C_4.
$C_4 = z_0$
$C_3 = \dot{z}_0 - \frac{g}{c}$ remember that $\dot{z}_0 = 0$ and rewrite $z(t)$

$z\,(t) = -\frac{g}{c^2} e^{-c\,t} - \frac{g + c\,v_{\text{air}}}{c}\,t + z_0$

3)

We simply switch the force from the fan to the y component. This gives the following.

$\ddot{x} = -c\,\dot{x}$
$\ddot{z} = -g - c\,\dot{z}$
$\ddot{y} = -c\,\dot{y} - c\,v_{\text{air}}$