Example Test 2

Justin's Version

CO_2 Molecule

A CO_2 molecule can be modeled as three masses connected by 2 springs of spring constant k. Let the mass of the carbon atom be $m_c = 3\,m$ and the mass of the oxygen atoms each be $m_O = 4\,m$ (even though these numbers may not look right, the ratio is correct). Allow only motions along the molecular axis (back and forth, not up and down).

a) Find the Lagrangian of this system.

b) Find the equations of motion of the system.

c) Find the Hamiltonian of the system.

d) Find the frequencies of the system.

e) Find the normal modes of the system. Are all these vibrations?

■ Solution

a)

$L = T - V$

$\quad = \frac{1}{2}\left(m_O \dot{x}_1^2 + m_c \dot{x}_2^2 + m_O \dot{x}_3^2\right) - \frac{k}{2}\left((x_2 - x_3)^2 + (x_3 - x_2)^2\right)$

b)

Use $\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$ to find the following.

$m_O \ddot{x}_1 + k x_1 - k x_2 = 0$

$-k x_1 + m_c \ddot{x}_2 + 2 k x_2 - k x_3 = 0$

$-k x_2 + m_o \ddot{x}_3 + k x_3 = 0$

c)

$H = T + V$

$\quad = \frac{1}{2}\left(m_O \dot{x}_1^2 + m_c \dot{x}_2^2 + m_O \dot{x}_3^2\right) + \frac{k}{2}\left((x_2 - x_3)^2 + (x_3 - x_2)^2\right)$

Put this into the form $H(q_i, p_i)$.

$H = \frac{1}{2}\left(\frac{p_1^2}{m_o} + \frac{p_2^2}{m_c} + \frac{p_3^2}{m_o}\right) + \frac{k}{2}\left((x_2 - x_3)^2 + (x_3 - x_2)^2\right)$

d)

Place the equations of motion in a matrix. Also assume that the motions are periodic.

$$\begin{pmatrix} k - m_o \omega^2 & -k & 0 \\ -k & 2k - m_c \omega^2 & -k \\ 0 & -k & k - m_o \omega^2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

Giving the following secular equation.

$$\text{Det}\left[\begin{pmatrix} k - m_o \omega^2 & -k & 0 \\ -k & 2k - m_c \omega^2 & -k \\ 0 & -k & k - m_o \omega^2 \end{pmatrix}\right]$$

$$-k^2 \omega^2 m_c - 2 k^2 \omega^2 m_o + 2 k \omega^4 m_c m_o + 2 k \omega^4 m_o^2 - \omega^6 m_c m_o^2$$

Solve the secular equation for ω^2. This gives 3 eigenvalues. If you get confused how to solve a cubic equation look to your mathematical handbook. The other trick is to notice that all terms have at least ω^2 in them. That means $\omega = 0$ is clearly an eigenvalue. You can divide by ω^2 and have a quadratic you can solve.

$\omega_1 = 0$

$\omega_2 = \sqrt{\frac{k}{m_o}}$

$\omega_3 = \sqrt{\frac{k}{m_o} + 2\,\frac{k}{m_c}}$

e)
Any mode with zero frequency is suspect. Plug into our earlier matrix to see that $a_1,\ a_2,\ a_3 = 0$.
Try ω_2

$$\begin{pmatrix} k - m_o\,\frac{k}{m_o} & -k & 0 \\ -k & 2\,k - m_c\,\frac{k}{m_o} & -k \\ 0 & -k & k - m_o\,\frac{k}{m_o} \end{pmatrix} . \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$\mathbf{MatrixForm}\left[\begin{pmatrix} k - m_o\,\frac{k}{m_o} & -k & 0 \\ -k & 2\,k - m_c\,\frac{k}{m_o} & -k \\ 0 & -k & k - m_o\,\frac{k}{m_o} \end{pmatrix} . \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right]$$

$$\begin{pmatrix} -k\,a_2 \\ -k\,a_1 - k\,a_3 + a_2\left(2\,k - \frac{k\,m_c}{m_o}\right) \\ -k\,a_2 \end{pmatrix}$$

For all elements of this array to equal zero $a_2 = 0$. From this you quickly see that $a_1 = -a_2$. Spectroscopists will call this an antisymmetric mode.

Now look for ω_3.

$$\mathbf{MatrixForm}\left[\begin{pmatrix} k - m_o\left(\frac{k}{m_o} + 2\,\frac{k}{m_c}\right) & -k & 0 \\ -k & 2\,k - m_c\left(\frac{k}{m_o} + 2\,\frac{k}{m_c}\right) & -k \\ 0 & -k & k - m_o\left(\frac{k}{m_o} + 2\,\frac{k}{m_c}\right) \end{pmatrix} . \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\right]$$

$$\begin{pmatrix} -k\,a_2 + a_1\left(k - m_o\left[\frac{2\,k}{m_c} + \frac{k}{m_o}\right]\right) \\ -k\,a_1 - k\,a_3 + a_2\left(2\,k - m_c\left[\frac{2\,k}{m_c} + \frac{k}{m_o}\right]\right) \\ -k\,a_2 + a_3\left(k - \left(\frac{2\,k}{m_c} + \frac{k}{m_o}\right)m_o\right) \end{pmatrix}$$

From the first and third element we see that $a_1 = a_3$, else they could not be equal zero. More algebra will show that $a_2 = -a_1(m_o/m_c)$.

Stability

A particle of mass m moves in the following potential. Is there a stable orbit? If so, what is the frequency, ω, of the small oscillatory motion?

1) $V = k\,x^{-2}\,e^{x} + c$

2) $V = A\,\mathrm{Cos}(k\,x + \pi/4)$

Solution

1)
Take the first derivative to see if there is a min or max.

$$\frac{k_2\,e^x}{x^2} - \frac{2\,k_2\,e^x}{x^3}$$

A bit of algebra later and we see that the first derivative has a zero at $x = 2$. Look to the second derivative to see if it is a max or min.

$$\frac{6\,k_2\,e^x}{x^4} - \frac{4\,k_2\,e^x}{x^3} + \frac{k_2\,e^x}{x^2}$$

At $x = 2$ this is positive, showing a minimum. We can find the effective spring constant by looking at the value of the second derivative.
We do this and see the following.
$k_{\mathrm{eff}} = V'' = \frac{e^2\,k}{8}$
Frequency follows from this.
$\omega = \sqrt{\frac{k_{\mathrm{eff}}}{m}}$

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Plot[x^-2 e^x, {x, -10, 10}]
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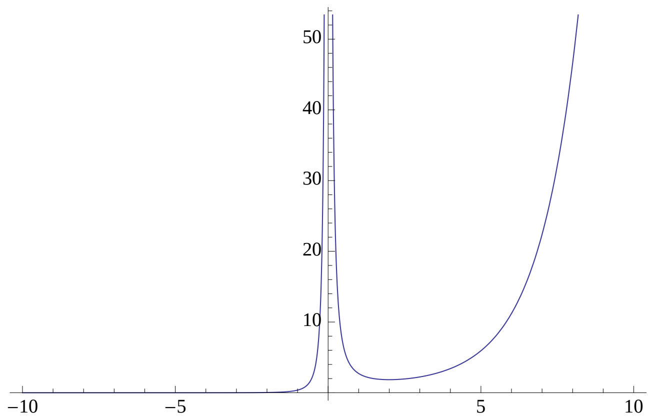


2)
Zeros for this are easy to find. Were the function simply Cos(x) the zeros would be at $x = 0, \pi/k, 2\pi/k, n\pi/k$. For this function we shift by $\pi/4$, giving minima at $x = -\pi/4k, 3\pi/4k, (4n-1)/4k$. Only half of these are minima (3π/4k). The others are maxima.
The frequency can be found the same way as before.

$$k_{\text{eff}} = A k^2$$

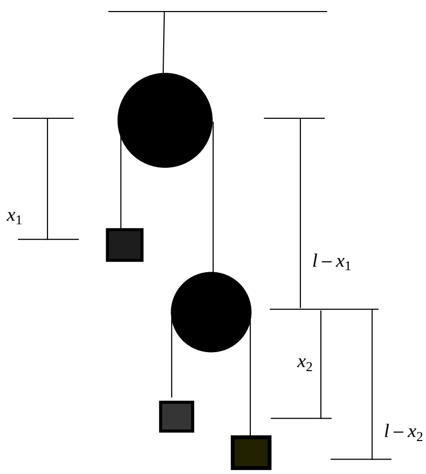
$$\omega = \sqrt{\frac{k_{\text{eff}}}{m}}$$

Double Atwood's Machine.

The following diagram shows a double Atwood's machine. Let each pulley be of mass M and radius r, each block be of mass m_i, and each string be of length l.

1)
Use the Lagrangian to find the equations of motion for this system.

2)
Finding a Hamiltonian in terms of q_i and p_i is difficult for the double Atwood's Machine. Why might that be?



Solution

1)

$$T = \frac{1}{2}\left(m_1\,\dot{x}_1^2 + m_2\,(\dot{x}_2 - \dot{x}_1)^2 + m_3\left(-\dot{x}_2 - \dot{x}_1\right) + M\,(-\dot{x}_1)^2\; +\; I\,\frac{\dot{x}_1^2}{a^2} + I\,\frac{\dot{x}_2^2}{a^2}\right)$$

$$V = -m_1\,g\,x_1 - M\,g(l - x_1) - m_2\,g\,(l - x_1 + x_2) - m_3\,g(l - x_1 + l - x_2)$$

$$L = \frac{1}{2}\left(m_1\,\dot{x}_1^2 + m_2\,(\dot{x}_2 - \dot{x}_1)^2 + m_3\,(-\dot{x}_2 - \dot{x}_1)^2 + M\,(-\dot{x}_1)^2\; +\; I\,\frac{\dot{x}_1^2}{a^2} + I\,\frac{\dot{x}_2^2}{a^2}\right) + m_1\,g\,x_1 + M\,g(l - x_1) + m_2\,g\,(l - x_1 + x_2) + m_3\,g(l - x_1 + l - x_2)$$

Use $\frac{\partial L}{\partial q} = \frac{d}{dt}\,\frac{\partial L}{\partial \dot{q}}$ to find the equations of motion.

x_1 :

$$(m_1 - M - m_2 - m_3)\,g = m_1\,\ddot{x}_1 + m_2\left(\ddot{x}_1 - \ddot{x}_2\right)\; + m_3\left(\ddot{x}_2 + \ddot{x}_1\right) + M\,\ddot{x}_1 + I\,\frac{\ddot{x}_1}{a^2}$$

x_2 :

$$m_2\,g - m_3\,g = m_2\left(\ddot{x}_2 - \ddot{x}_1\right) + m_3\left(\ddot{x}_2 + \ddot{x}_1\right) + I\,\frac{\ddot{x}_2}{a^2}$$

2)

While T is easy to write in terms of velocities, for this case it is harder to write in terms of momenta. $\frac{\partial L}{\partial \dot{x}_1} = p_1$ is not simply $m_1\,\dot{x}_1$ as in other problems we have worked.

Forces of Constraint

A small, wet bar of soap of mass m can move about the inside of a hemispherical bowl of radius R.

1) Write the Lagrangian for this system.

2) What is the constraint for this system.

3) Write the equations of motion.

4) What is the normal force on the soap?

Solution

1)
$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - m\,g\,z$
$\quad = \frac{m}{2}\left(\dot{r}^2 + r^2\,\dot{\theta}^2 + r^2\,\dot{\phi}^2\,\sin(\theta)\right) - m\,g\,r(1 - \cos(\theta))$
ϕ must be constant, so it may be ignored.

2)
The constraint it for the soap to only move along the hemisphere.
$r = R\,;\ \dot{r} = 0$
$f = r - R$

3)
$L = \frac{m}{2}\left(\dot{r}^2 + r^2\,\dot{\theta}^2\right) - m\,g\,r(1 - \cos(\theta))$

r :
$m\,g\cos(\theta) + m\,r\,\dot{\theta}^2 + m\,\ddot{r} + \lambda_r = 0$
we know that $\ddot{r} = 0$. So we can simplify
$m\,g\cos(\theta) + m\,r\,\dot{\theta}^2 = \lambda_r$

θ :
$-m\,g\sin(\theta) + m\,r^2\,\ddot{\theta} = 0$
or
$\ddot{\theta} = \frac{g}{r^2}\sin(\theta)$

4)
The normal force is already done in part 3.
$F_N = \lambda_r\,\frac{\partial f}{\partial r} = Q_r = m\,g\cos(\theta) + m\,r\,\dot{\theta}^2$