Justin' s Practice Final Exam

Balloon Boy (similar to 7.25)

A small child climbs into a large silver balloon. He does so carrying several bags of sand of mass m_0. The balloon, boy, sand system is neutrally buoyant when the child begins slowly releasing sand from the bags at a constant rate. The boy and balloon's combined mass (no sand) is M.
Find the height of the balloon and its velocity when all the sand has been released. Assume that the upward buoyancy force remains constant. Neglect air resistance.

Solution

Identify forces. A bit about my notation. $m_0$is the initial mass of the sand bags. m is the time varying mass of sand in the bags.
$F_B - (M + m_0)\, g = 0$

$m = m_0(1 - r\,t)$

$F_B - (M + m)\, g = (M + m)\ \frac{d\mathbf{v}}{d\,t}$

Solve to find $\frac{\mathrm{dv}}{\mathrm{dt}}$.
$(M + m_0)\, g - (M + m)\, g = (M + m)\ \frac{\mathrm{dv}}{\mathrm{dt}}$

$$\frac{\mathrm{dv}}{\mathrm{dt}} = \frac{(m_0 - m)}{(M+m)}\, g$$
$$= \quad \frac{m_0\, r\, t}{M + m_0 - m_0\, r\, t}\, g$$

Notice the solution to the following integral.

$$\int \frac{t}{(a - b\,t)}\, dt$$

$$-\frac{t}{b} - \frac{a\,\mathrm{Log}[-a + b\,t]}{b^2}$$

This gives us the velocity

$$v = -g\,t - \frac{(M + m_0)\ln(m_0\, r\, t - M - m_0)}{m_0\, r}\, g$$

Integrate again to find y(t).

$$\int \mathrm{Log}[a\,t - b]\, dt$$

$$-t - \frac{b\,\mathrm{Log}[-b + a\,t]}{a} + t\,\mathrm{Log}[-b + a\,t]$$

This gives the following.

$$y = -\text{gt}^2 + \frac{(M+m_0)\, g}{m_0\, r} \left(t + \frac{(M+m_0)\ln(M+m_0-m_0\, r\, t)}{m_{0\, r}} - t\ln(M+m_0-m_0\, r\, t)\right)$$

Set $t = \frac{1}{r}$ to find y and v when the sand runs out.

Cut a Frisbee

Your jerk friend cut your Frisbee in half. Pretend that the Frisbee is a flat disc of constant mass density ρ.
a) Where is the moment of inertia of your cut Frisbee?
b) What is the moment of inertia of this cut disc about the point that was it's center before the cut?
c) If you hung the half disc from the center of it's flat side, what would be it's period of oscillation?

■ Solution

a)

$$z_{\text{cm}} = \frac{\int_0^a \int_0^\pi \rho(r \sin\theta)\, r\, dr\, d\theta}{\int_0^\pi \rho\, r\, dr\, d\theta}$$

$$\textbf{top} = \int_0^a \int_0^\pi \rho\, (r\, \text{Sin}[\theta])\, r\, d\theta\, dr$$

$$\frac{2\, a^3\, \rho}{3}$$

$$\textbf{bottom} = \int_0^a \int_0^\pi \rho\, r\, d\theta\, dr$$

$$\frac{1}{2}\, a^2\, \pi\, \rho$$

We take the ratio and get the distance from the origin of our center of mass.

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top / bottom
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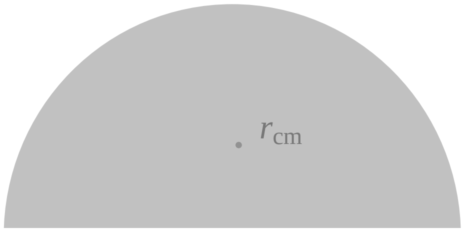
$$\frac{4\, a}{3\, \pi}$$



b)

If it was a whole disc the moment if inertia would be $I = \frac{m r^2}{2} = \pi \frac{\rho r^4}{2}$.

Since it is cut in half it has half the moment. Thus,

$I = \frac{\pi \rho r^4}{4}$=.

You could find the same thing by integrating.

$$\int_0^a \int_0^\pi \rho\, r\, r^2\, d\theta\, dr$$

$$\frac{1}{4} a^4 \pi \rho$$

c)

This part is easy. Just plug in the formula.

$$T = 2\pi \sqrt{\frac{I}{m g l}} = 2\pi \sqrt{\frac{\pi \rho r^4}{4\left(\frac{1}{2}\right)\pi r^2 \rho g \left(\frac{4r}{3\pi}\right)}} = 2\pi \sqrt{\frac{3\pi r}{8 g}}$$

Dark Matter

The best evidence for Dark Matter is that the rotational speed of objects as a function of distance from a galaxy's center is different from the expected, given the amount of known luminous (regular, non-dark) matter. Assume a galaxy is and spherical and that it's mass distribution ρ goes as $\rho = \rho_0 r^{-2}$(this function isn't how real galaxies work, but we'll pretend.

a) Write the gravitational potential as a function of radius. As you've done before, you can think about this as you would Gauss' law

b) What is the rotational speed of an object in the galaxy as a function of radius?

Note: If this isn't the observed velocity distribution then there is evidence of dark matter (in this made up galaxy).

Solution

a)
Yes, galaxies aren't spherical. We're going to pretend to keep the math a little easier.
$g = -G \frac{M_{enc}}{r^2}$
$M_{enc} = \int (\rho_0 r^{-2}) 4 \pi r^2 \, dr$
This happens to be easy to integrate.
$M_{enc} = 4 \pi \rho_0 r$
Thus,
$g = -G \frac{4 \pi \rho_0}{r}$

Find the potential from the field.
$\Phi = -\int g \, dr = \mathrm{G} 4\pi \rho_0 \ln(r)$

b)
Some mass m will experience a centripetal force from gravity. This is an easy thing to do since we decided the motion was circular.
$F_{cent} = m \frac{v^2}{r}$
Set this equal to the force from gravity.
$G \frac{m 4 \pi \rho_0}{r} = m \frac{v^2}{r}$
$v = \sqrt{4 G \pi \rho_0}$

Loop de Loop

A roller coaster takes its riders in a vertical circle (a loop de loop). We'll treat the roller coater car as a point object of mass m.
a) Write the Lagrangian for this system.
b) Use Lagrange multipliers to find the normal force.
c) People start to pass out with normal forces around 3g. If the speed at the bottom is 25m/s and the turn is 20m tall will people pass out? Does the cart have enough kinetic energy to actually get to the top of the loop?

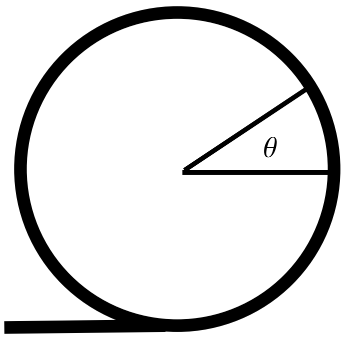
Note: Real roller coasters don't use a circle for this so the accelerations will be more gentle.

Solution

$T = \frac{1}{2} m v^2 = \frac{1}{2} m \left(r^2 \dot{\theta}^2 + \dot{r}^2\right)$

$V = m g r \sin(\theta)$

In this case θ is measured from the horizontal. You could do this differently, but you would get a different functional form.



$L = T - V = \frac{1}{2} m\left(r^2 \dot{\theta}^2 + \ \dot{r}^2\right) - m g r \sin(\theta)$

b)
Constraint function is the following.
$f = R - r = 0$

Now solve Lagrange's equation give the following.
θ equations first.
$-m g r \cos(\theta) - m r^2 \ddot{\theta} = 0$
r equation
$m r \dot{\theta}^2 - m g \sin(\theta) \ - m \ddot{r} - \lambda = 0 \qquad \ddot{r} = 0$

Normal force is the following.
$\lambda = m r \dot{\theta}^2 - m g \sin(\theta)$

c)
Notice we can rewrite the normal force as the following.
$N = m \frac{v^2}{r} - m g \sin(\theta)$
$a = \frac{v^2}{r} - g \sin(-90)$

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25^2/20 + 9.8
41.05
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Divide by g to figure out how many "g" forces there are.

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41 / 9.8
4.18367
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Looks like people pass out.

d)
Compare kinetic energy and potential energy
First kinetic.

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.5 (25^2)
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312.5
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and Potential.

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9.8 (40)
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392.
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V>T
Nope. It never gets to the top. Circles aren't very good for roller coasters.

A Totally Uncreative Potential Question

A particle is in the potential $U = x^4 + 4\,x^3$.
a) What is the minimum of this potential?
b) Is the minimum stable?
c) If a particle is released from $x = 0$, what would be it's turning point?
d) What would be the frequency of small oscillations about the minimum for a particle of mass m=1kg.

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Manipulate[Plot[x^4 + b x^3, {x, -6, 4}], {b, -10, 10}]
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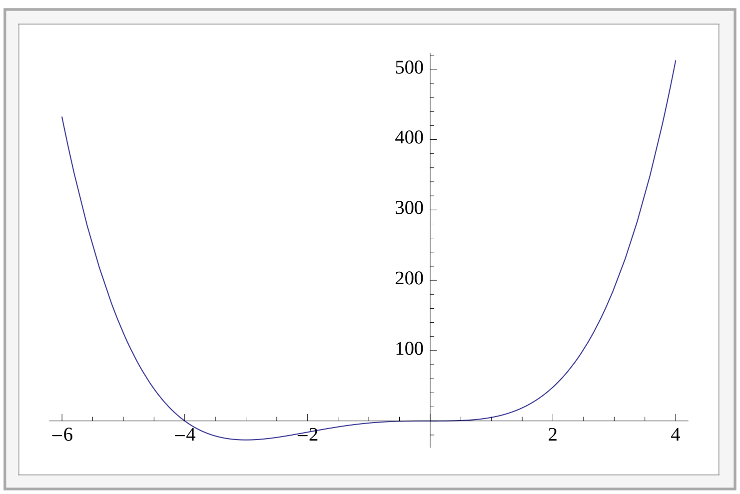

a)
Minimum of the potential
$\frac{dU}{dx} = 4\, x^3 + 12\, x^2 = 0$
$x^3 = -3\, x^2$
$x = -3$

b)
The minimum is stable.
$\frac{d^2 U}{dx^2} = 12\, x^2 + 24\, x$
$\frac{d^2 U}{dx^2}\,(-3) = 108 - 62 > 0$

c)
The energy at x=0 is zero. Where else is this true?
The two clear solutions to U=0 are x=0 and x=-4.
X=-4 is the other turning point.

d)
$k_{\text{eff}}\ \frac{d^2 U}{dx^2}\,(-3) = 108 - 62 = 46\ N/m$

Plug this in to get the frequency.
$\omega\ =\ \sqrt{\frac{k_{\text{eff}}}{m}}\ =\sqrt{\frac{46}{1}}$ Hz