Kinematics

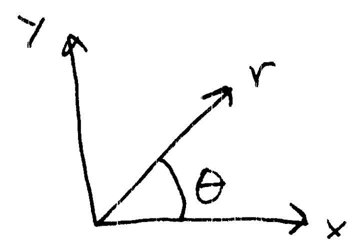
Cartesian Coordinates

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{x}\,\hat{x} + \dot{y}\,\hat{y} + \dot{z}\,\hat{z}$$ velocity

$$\vec{r} = (x, y, z)$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \ddot{x}\,\hat{x} + \ddot{y}\,\hat{y} + \ddot{z}\,\hat{z}$$ acceleration

Plane Polar

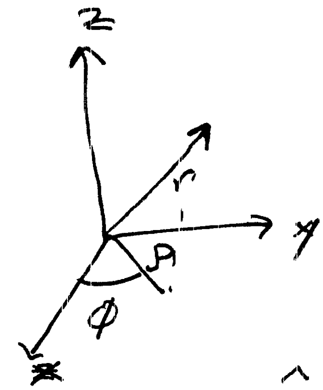


$$\vec{r} = r\,\hat{r}$$

$$\vec{v} = \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta}$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\theta}$$

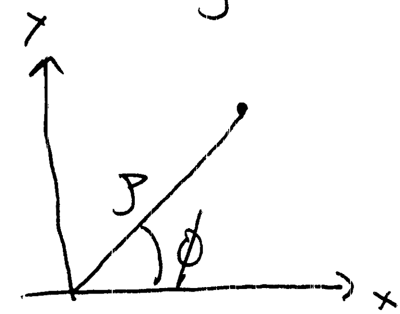
Cylindrical



$$\vec{r} = \rho\,\hat{\rho} + z\,\hat{z}$$

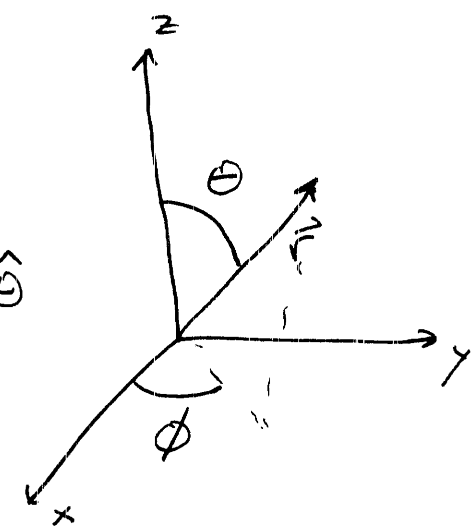
$$\vec{v} = \dot{\rho}\,\hat{\rho} + \rho\dot{\phi}\,\hat{\phi} + \dot{z}\,\hat{z}$$

$$\hat{\rho} \times \hat{\phi} = \hat{z}$$



$$\vec{a} = (\ddot{\rho} - \rho\dot{\phi}^2)\,\hat{\rho}$$
$$+ (2\dot{\rho}\dot{\phi} + \rho\ddot{\phi})\,\hat{\phi}$$
$$+ \ddot{z}\,\hat{z}$$

Spherical



$$\vec{r} = r\,\hat{r}$$

$$\vec{v} = \dot{r}\,\hat{r} + r\dot{\phi}\sin\theta\,\hat{\phi} + r\dot{\theta}\,\hat{\theta}$$

$$\vec{a} = (\ddot{r} - r\dot{\phi}^2\sin^2\theta - r\dot{\theta}^2)\,\hat{r}$$

$$\hat{r} \times \hat{\theta} = \hat{\phi}$$

$$+ (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\,\hat{\theta}$$

no dot

$$+ (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\,\hat{\phi}$$

What do these expressions mean?

In any coordinate systems, the location of the particle is given by the displacement from the origin.

$$\vec{r}(x, y, z) \qquad \vec{r}(\rho, \phi, \hat{z}) \qquad \vec{r}(r, \theta, \phi)$$

As the position of the particle changes, the variables making up the position vector change. For example, $r(t)$, $\theta(t)$, $\phi(t)$ in spherical coordinates. If we know how these functions change, we can use the expressions on the previous pages to find $\vec{r}$, $\vec{v}$, and $\vec{a}$.

Example: Suppose we are given a particle that experience a constant force F_0 toward the origin and has initial velocity $\vec{v} = v_0 \hat{\phi}$. Express the ~~trajectory~~ EOM in cylindrical coordinates.

$$\vec{F} = -F_0 \hat{\rho}$$

$$\vec{F} = m\vec{a} = (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\rho}$$
$$+(2\dot{\rho}\dot{\phi} + \rho\ddot{\phi})\hat{\phi} + \ddot{z}\hat{z}$$

EOM

$$-F_0 = \ddot{\rho} - \rho\dot{\phi}^2$$

$$0 = 2\dot{\rho}\dot{\phi} + \rho\ddot{\phi}$$

We will find a more robust solution method, but if F_0 points inward, try $\dot{\phi} = \omega$ = constant

$\rho = \rho_0$ = constant.

Both equations are satisfied with

$$-F_0 = -\rho_0 \omega^2$$

Exact Differential Forms

Suppose we have a force $\vec{F}$ s.t. $\nabla \times \vec{F} = 0$, we know a potential function exists. How do we find it?

$$dW = \vec{F} \cdot d\vec{r} = -\frac{\partial V}{\partial x} dx - \frac{\partial V}{\partial y} dy - \frac{\partial V}{\partial z} dz$$

which is an exact differential form. Cal III should taught you how to find V given the partials but an example might help.

Ex Let $V = xyz + 2y^2$

$$\vec{F} = -yz\hat{x} - (xz + 4y)\hat{y} - xy\hat{z}$$

Now work backwards to find V

$$-yz = -\frac{\partial V}{\partial x} = F_x$$

$$V = \int yz\,dx + f(y,z)$$

$$V = xyz + f(y,z)$$

y-component

$$-\frac{\partial V}{\partial y} = -(xz + 4y) = -xy - \frac{\partial f}{\partial y} = F_y$$

$$\frac{\partial f}{\partial y} = 4y$$

$$f(y,z) = \int 4y\,dy + g(z)$$

$$= 2y^2 + g(z)$$

$$V = xyz + 2y^2 + g(z)$$

$$-\frac{\partial V}{\partial z} = -xy - \frac{\partial g}{\partial z} = -xy = F_z$$

$$\frac{\partial g}{\partial z} = 0 \qquad g = C$$

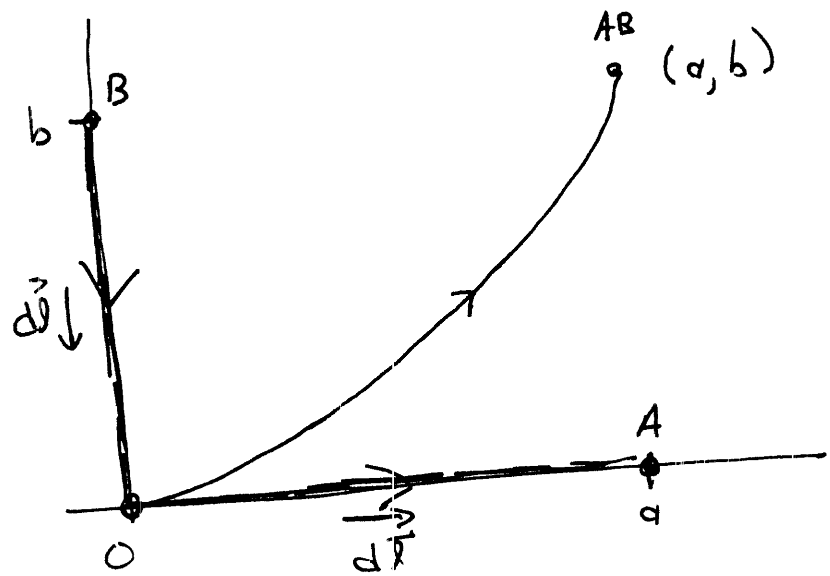
$$V = xyz + 2y^2 + C$$

Path integrals

The one we care about is

$$W = \int_{A \to B} \vec{F} \cdot d\vec{l}$$

Let $\vec{F} = x\hat{x} + y\hat{y}$. Evaluate $\int \vec{F} \cdot d\vec{l}$ along three paths $O = (0,0)$ to $A = (a,0)$ in straight line, $B = (0,b)$ to $O = (0,0)$ along line, and $O = (0,0)$ to $AB = (a,b)$ along $y = \frac{b}{a^2}x^2$.



Path 1 $\quad O \to A \quad$ work positive

$d\vec{l} = dx\,\hat{x} \quad$ from $0 \to a \quad$ dx positive

$$W = \int_{O\to A} \vec{F}\cdot d\vec{l} = \int_0^a (x\hat{x} + y\hat{y})\cdot(\hat{x}\,dx)$$

$$= \int_0^a x\,dx = \frac{a^2}{2}$$

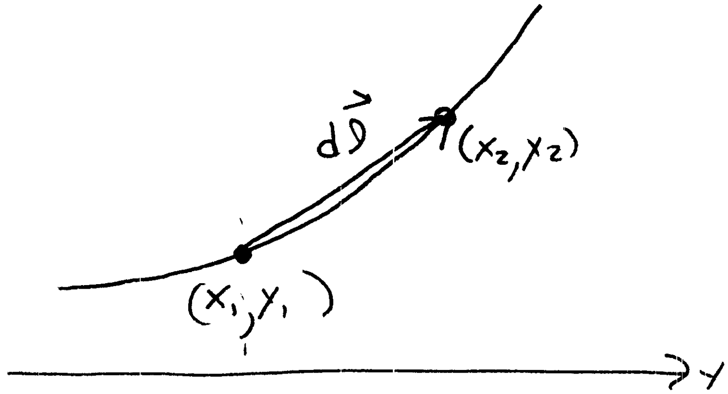
Path 2 $\quad B \to O \quad$ work negative

$d\vec{l} = dy\,\hat{y} \quad$ from $b \to 0 \quad$ dy negative

$$W = \int_{B\to O} \vec{F}\cdot d\vec{l} = \int_b^0 (x\hat{x} + y\hat{y})\cdot(\hat{y}\,dy)$$

$$= \int_b^0 y\,dy = -\frac{b^2}{2}$$

Path 3



$$d\vec{l} = (x_2 - x_1,\ y_2 - y_1) \qquad \frac{dy}{dx} \sim \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \left(\Delta x,\ \frac{dy}{dx}\Delta x\right)$$

$$= \left(1,\ \frac{dy}{dx}\right) dx = \left(\hat{x} + \frac{dy}{dx}\hat{y}\right) dx$$

$$W = \int_{O \to AB} \vec{F} \cdot d\vec{l} = \int_0^a (x\hat{x} + y\hat{y}) \cdot \left(\hat{x} + \frac{dy}{dx}\hat{y}\right) dx$$

$$= \int_0^a \left(x + y\frac{dy}{dx}\right) dx \qquad y = \frac{b}{a^2}x^2, \quad \frac{dy}{dx} = \frac{2b}{a^2}x$$

$$= \int_0^a \left(x + 2\frac{b^2}{a^4}x^3\right) dx$$

$$= \frac{a^2}{2} + \frac{1}{2}\frac{b^2}{a^4} \cdot a^4 = \frac{1}{2}(a^2 + b^2)$$

Note we still have a problem with the sign of dx. I recommend integrating from $0 \to a$ and switching the sign if the actual integral was from $a \to 0$.