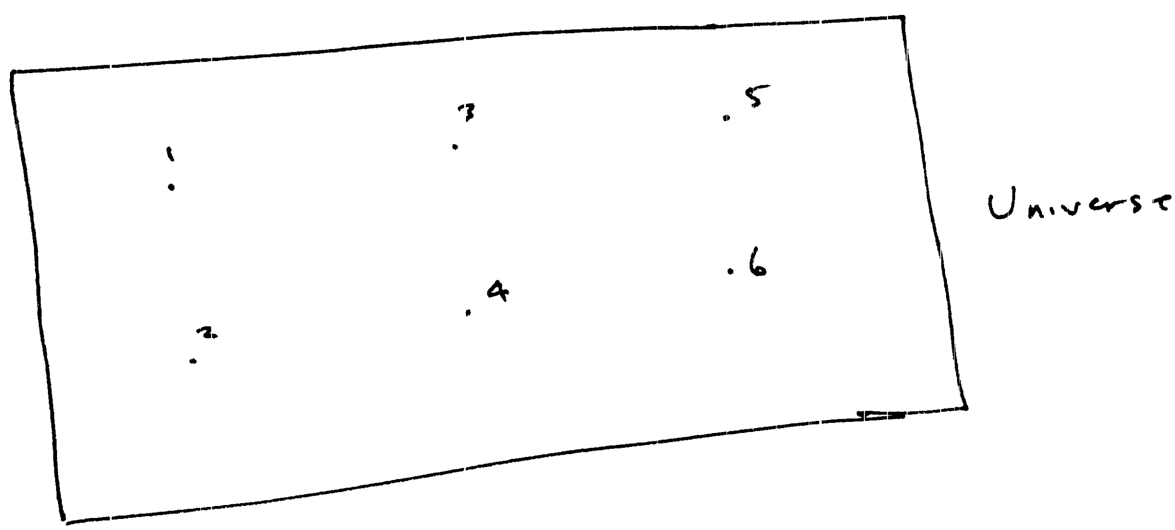
Variable Mass (Rocket) Problems

We have been considering systems where the N particles in the system are constant.

We will now consider the case where the number of particles in the system can change.



If the above is a gas of six particles, then the total momentum is about zero.

$$\underset{\text{System}}{\vec{P}_{cm}} = 0 \qquad \underset{\text{Environment}}{\vec{P}_{env}} = 0$$

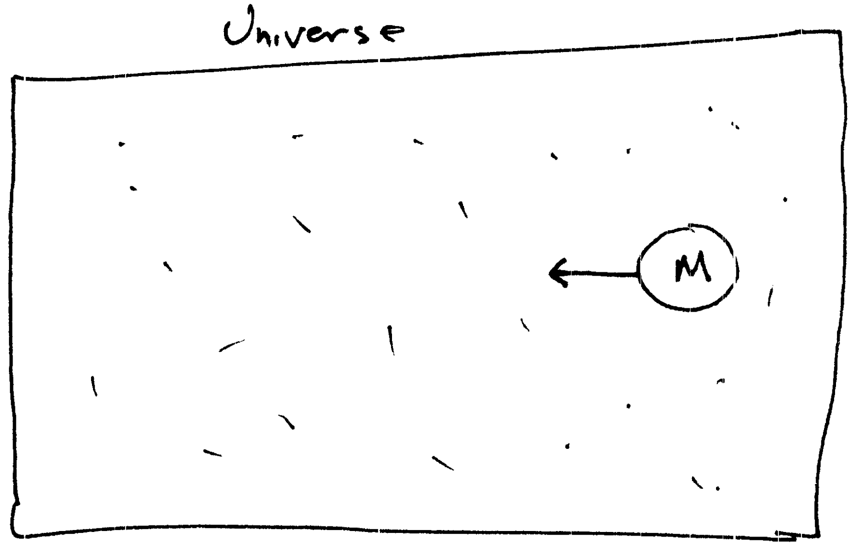
Now suppose in the time Δt we redefine the system to only include particles traveling to the right.

The total momentum of the universe is still zero $\vec{P}_{cm} + \vec{P}_{env} = 0$, but both $\vec{P}_{cm}$ and $\vec{P}_{env}$ are non-zero.

Redefining the system changed the momentum of the system and therefore produced a "force" on the center of mass.

$$\vec{F}_{cm} = \frac{\Delta \vec{p}}{\Delta t} = \frac{\vec{P}_{cm}}{\Delta t} = \frac{-\vec{P}_{env}}{\Delta t}$$

Let's try a different example, a rock moving through a stationary sticky cloud of dust.



If the velocity of the rock at $t=0$ is v_0, then the total momentum of the universe is Mv_0, which must be conserved.

Let's assume the rate dust is accumulated is proportional to the speed of the system,

$$\alpha v = \frac{dm}{dt}$$

Let's separate the momentum of the universe into the momentum of the dust/rock system $\vec{p}_{sys}$ and other stuff, the environment, $\vec{p}_{env}$.

$$\vec{p}_{universe} = \vec{p}_{sys} + \vec{p}_{env}$$

$$\frac{d\vec{p}_{uni}}{dt} = 0 = \frac{d\vec{p}_{sys}}{dt} + \frac{d\vec{p}_{env}}{dT}$$

$$\vec{p}_{sys} = M_{sys}\vec{v}_{sys}$$

$$\vec{p}_{env} = M_{env}\vec{v}_{env}$$

$$0 = M_{env}\frac{d\vec{V}_{env}}{dt} + \vec{V}_{env}\frac{dM_{env}}{dt}$$

$$+ M_{sys}\frac{d\vec{V}_{sys}}{dt} + \vec{V}_{sys}\frac{dM_{env}}{dt}$$

Ignore external forces on the environment, $\dot{\vec{V}}_{env} = 0$.

$$\boxed{M_{sys}\frac{d\vec{V}_{sys}}{dt} = -\vec{V}_{env}\frac{dM_{env}}{dt} - \vec{V}_{sys}\frac{dM_{sys}}{dt}}$$

For our rock/dust system $\vec{V}_{env} = 0$, and let
$M_{sys} \equiv m$, $\vec{V}_{sys} \equiv v$

$$m\frac{dv}{dt} = -v\frac{dm}{dt}$$

With our assumption that $\frac{dm}{dt} = \alpha v$ this becomes

$$m\frac{dv}{dt} = -\alpha v^2$$

The dust attaching to the system exerts a quadratic drag force. If the initial velocity at $t=0$ is v_0.

$$\int_{v_0}^{v} \frac{dv}{v^2} = -\int_0^t \frac{\alpha}{m}\, dt$$

$$-\frac{1}{v}\Big|_{v_0}^{v} = -\frac{\alpha}{m} t$$

$$\frac{1}{v_0} - \frac{1}{v} = -\frac{\alpha}{m} t$$

$$\frac{1}{v} = \frac{1}{v_0} + \frac{\alpha t}{m} = \frac{m + \alpha t v_0}{v_0 m}$$

$$v = \frac{v_0 m}{m + \alpha t v_0} = v_0 \left(\frac{1}{1 + \alpha t v_0/m} \right)$$

What happens if the dust is moving at the same rate as the rock, but still accreting at the same rate?

$$\vec{V}_{sys} = \vec{V}_{env}$$

$$m \frac{dv_{sys}}{dT} = -\vec{V}_{sys} \underbrace{\left(\frac{dM_{env}}{dt} + \frac{dM_{sys}}{dt} \right)}_{=0 \text{ if mass conserved}}$$

$\Longrightarrow$ No force

$\Longrightarrow$ Force depends on relative velocity of system and the mass added to the system.

In general, with $m = M_{sys}$

$$\frac{dm}{dt} = \frac{dM_{sys}}{dt} = -\frac{dM_{env}}{dT}$$

$$m \frac{d\vec{v}_{sys}}{dt} = -\vec{V}_{env} \left(-\frac{dm}{dt} \right) - \vec{V}_{sys} \frac{dm}{dt}$$

$$m(t)\,\frac{d\vec{v}_{sys}}{dt} = \left(\vec{v}_{env} - \vec{v}_{sys}\right)\frac{dm}{dt}$$

Let $\vec{v}_{rel} = \vec{v}_{env} - \vec{v}_{sys} =$ relative velocity

$$m(t)\,\frac{d\vec{v}_{sys}}{dt} = \vec{v}_{rel}\,\frac{dm}{dT}$$

Naturally, we can also exert an external force on the object. So in general, for a system of changing mass.

$$m(t)\,\frac{d\vec{v}_{sys}}{dt} = \vec{v}_{rel}\,\frac{dm}{dt} + \vec{F}_{ext}$$

<u>Ex</u> Consider a rocket, where the rocket is propelled with a constant exhaust velocity (relative to the rocket) of V_{rel}. The mass of the rocket changes as $m(t) = m_0 - \alpha t$ where m_0 is the initial mass. The rocket runs out of fuel at $m(T_b) = m_b$ where m_b is the burn-out mass and T_b is the time to burn out. How fast is the rocket going at T_b?

$$m(t)\frac{dv}{dt} = -V_{rel}\frac{dm}{dt}$$

exhaust gas ejected in - direction to direction of motion.

$$\frac{dm}{dt} = -\alpha$$

$$m(t)\frac{dv}{dt} = \alpha V_{rel}$$

$$dv = \frac{\alpha V_{rel}}{m(t)}\,dt$$

$$\int_0^{V_b} dv = \int_0^{T_b} \frac{\alpha v_{rel}}{m_0 - \alpha t}\, dt$$

$$V_b = -\frac{1}{\alpha}\, \alpha v_{rel} \ln\left(m_0 - \alpha t\right)\Big|_0^{T_b}$$

$$= \ln\left(\frac{m_0}{m_0 - \alpha T_b}\right) v_{rel}$$

$$m_0 - \alpha T_b = m_f$$

$$V_b = v_{rel} \ln\left(\frac{m_0}{m_f}\right)$$

Ex I had to do this every year, because my kid wanted a pinata for her birthday. Consider throwing a baseball of mass m_0 with a rope of mass λ per unit length straight up in the air. How high does it go if initial velocity is v_0?

Let the environment contain the part of the rope still on the ground. $V_{env} = 0$.

$$v_{rel} = \cancel{v_{sys} - v_{env}}$$

$$= v_{env} - v_{sys} = -v_{sys}$$

If the ball is thrown from a height h_0, the mass of the system is

$$m(y) = m_0 + \lambda y$$

$$m(y)\frac{d v_{sys}}{dt} = -v_{sys}\frac{dm}{dt} - mg$$

$$m(y)\ddot{y} = -\dot{y}(\lambda\dot{y}) - mg$$

$$(m_0 + \lambda y)\frac{d\dot{y}}{dy}\frac{dy}{dt} = -\lambda v^2 - mg$$

$$(m_0 + \lambda y)v\frac{dv}{dt} = -\lambda v^2 - mg$$

and we're stuck.