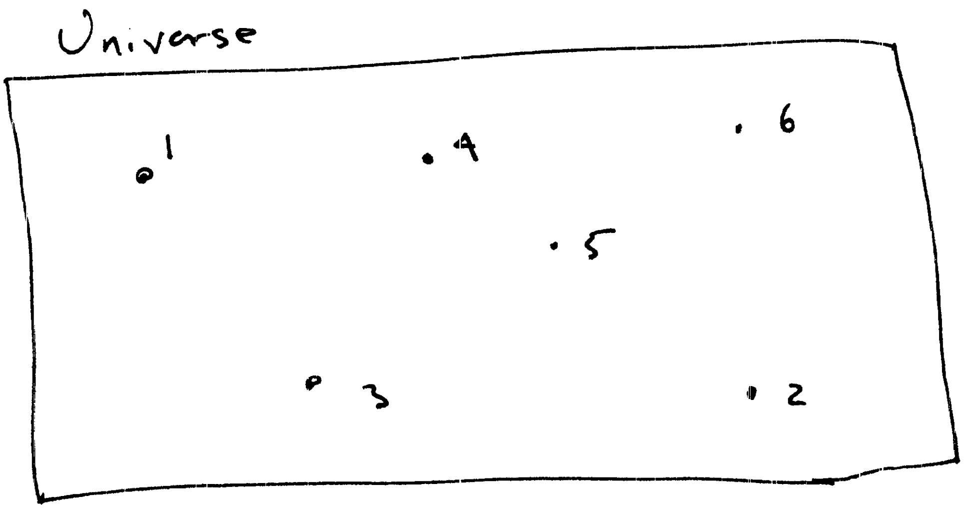
Systems of Particles

A system of particles is defined by telling which particles particles are in the system.

$\Rightarrow$ These particles do not need to be interacting, contiguous, or in any way related.



We could define odd particles as our system and then even particles would be the environment.

For the universe, energy, momentum, and angular momentum are conserved. There are no external forces.

Let our system of N particles have positions $\vec{r}_i$ and velocities $\vec{v}_i = \dot{\vec{r}}_i$ and masses m_i.

Total mass of system (M)

$$M = \sum_i m_i$$

Center of Mass $(\vec{r}_{cm})$

$$\vec{r}_{cm} = \frac{\sum m_i \vec{r}_i}{M}$$

$\Longrightarrow$ Average location of the mass.

Total Momentum $(\vec{P})$

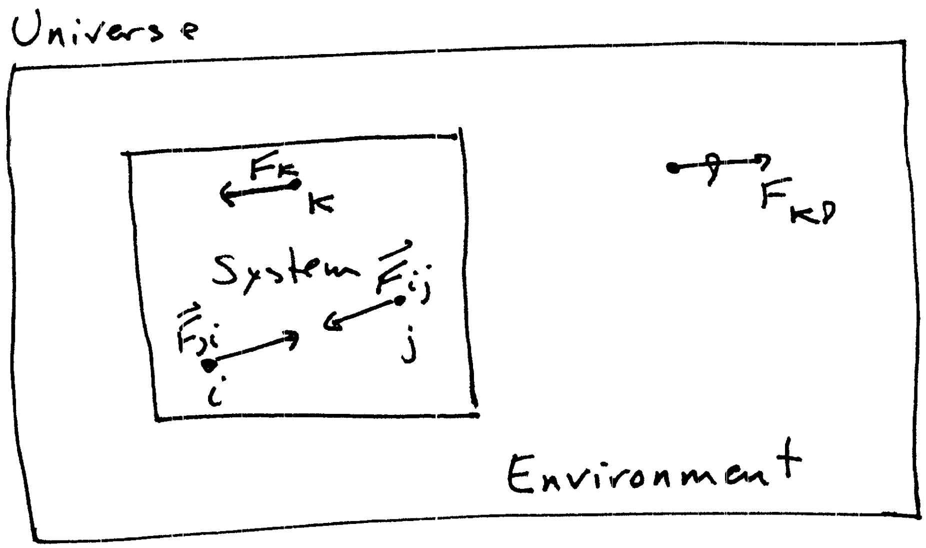
$$\vec{P} = \sum m_i \vec{v}_i = \sum \vec{p}_i$$

Velocity of Center of Mass

$$\vec{v}_{cm} = \dot{\vec{r}}_{cm} = \frac{1}{M} \sum m_i \dot{\vec{r}}_i = \frac{1}{M} \sum m_i \vec{v}_i$$

$$\vec{V}_{cm} = \frac{\vec{P}}{M}$$

The total momentum of the system is equal to that of a point mass of mass M moving with the velocity of the center of mass.



<u>Internal Forces</u> Forces between m_i, m_j in the system, $\vec{F}_{ij}$

<u>External Forces</u> Forces on particles in the system by particles outside the system.

The total external force is $\vec{F}_i$

Internal forces come is pairs (Newton III)

$$\vec{F}_{ij} = -\vec{F}_{ji}$$

<u>Newton II for particle i</u>

$$\dot{\vec{p}}_i = \underbrace{\vec{F}_i}_{\text{external}} + \underbrace{\sum_j \vec{F}_{ji}}_{\text{total internal}}$$

<u>Sum over particles in the system</u>

$$\sum_i \dot{\vec{p}}_i = \sum_i \vec{F}_i + \sum_{ij} \vec{F}_{ji}$$

Because of Newton III, the second term, the net force of the internal forces vanish.

$$\dot{\vec{P}} = \sum_i \dot{\vec{p}}_i = \sum \vec{F}_i = M\dot{\vec{v}}_{cm}$$

The acceleration of the center of mass is the same as the acceleration of a point mass with the total mass of the system M acted on by the sum of the forces $\sum_i \vec{F}_i$.

$\Rightarrow$ Note, in general the force will not act at the location of the CM but that's fine.

<u>Body Forces</u> - Forces that act equally on all particles in the system.

Example - Gravity (near earth)

$$\vec{F}_i = m_i \vec{g}$$

$$\sum_i \vec{F}_i = \sum_i m_i \vec{g} = M\vec{g} = M\dot{\vec{v}}_{cm}$$

$\Rightarrow$ Body forces act as if all the mass was at the center of mass.

If the total external force is zero, then

$$\dot{\vec{P}} = 0 \quad \text{and momentum is conserved.}$$

Angular Momentum $(\vec{L})$ about origin

$$\vec{L}_i = \vec{r}_i \times \vec{p}_i = \vec{r}_i \times m_i \vec{v}_i$$

$$= m_i (\vec{r}_i \times \vec{v}_i)$$

Total Angular Momentum

$$\vec{L} = \sum \vec{r}_i \times \vec{p}_i$$

Find the time rate of change of $\vec{L}$

$$\frac{d\vec{L}}{dt} = \sum_i \underbrace{\dot{\vec{r}}_i \times m_i \vec{v}_i}_{\vec{v}_i \times m\vec{v}_i = 0} + \vec{r}_i \times m_i \frac{d\vec{v}_i}{dt}$$

$$= \sum_i \vec{r}_i \times \dot{\vec{p}}_i$$

The time rate of change of particle i

$$\dot{\vec{p}}_i = \vec{F}_i + \sum_j \vec{F}_{ji}$$

$$\frac{d\vec{L}}{dt} = \sum \vec{r}_i \times \vec{F}_i + \sum_{ij} \vec{r}_i \times \vec{F}_{ji}$$

Look for force pairs in the second sum

$$\vec{r}_i \times \vec{F}_{ji} + \vec{r}_j \times \vec{F}_{ij}$$

$$\vec{F}_{ij} = -\vec{F}_{ji}$$

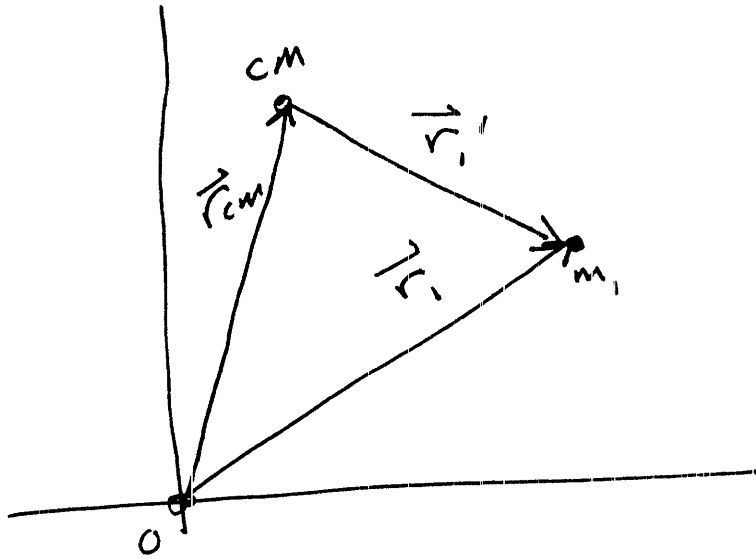
$$(\vec{r}_j - \vec{r}_i) \times \vec{F}_{ij} = 0$$

if forces central.

$$\frac{d\vec{L}}{dt} = \sum \vec{r}_i \times \vec{F}_i \equiv \vec{N} \quad \text{torque}$$

For an isolated system, $\vec{N} = 0$ and the total angular momentum is conserved.

Write particles location relative to CM



$$\vec{r}_i = \vec{r}_{cm} + \vec{r}_i' \qquad \vec{v}_i' = \dot{\vec{r}}_i'$$

$$\vec{v}_i = \vec{v}_{cm} + \vec{v}_i'$$

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i = \sum_i (\vec{r}_{cm} + \vec{r}_i') \times m_i (\vec{v}_{cm} + \vec{v}_i')$$

$$= \left(\sum_i m_i\right) \vec{r}_{cm} \times \vec{v}_{cm} + \vec{r}_{cm} \times \sum_i m_i \vec{v}_i'$$

$$+ \left(\sum m_i \vec{r}_i'\right) \times \vec{v}_{cm} + \sum \vec{r}_i' \times m_i \vec{v}_i'$$

$$\sum m_i \vec{r}_i{}' = 0$$

since $\displaystyle \sum m_i \vec{r}_i{}' + \underbrace{\sum m_i \vec{r}_{cm}}_{M\vec{r}_{cm}} = \underbrace{\sum m_i \vec{r}_i}_{M\vec{r}_{cm}}$

<u>Likewise</u> $\displaystyle \sum m_i \vec{v}_i{}' = 0$

$$\vec{L} = \underbrace{\vec{r}_{cm} \times \vec{p}_{cm}}_{\text{Angular momentum of CM about origin}} + \underbrace{\sum \vec{r}_i{}' + m_i \vec{v}_i{}'}_{\text{Angular momentum about CM.}}$$

Through a very similar calculation

Kinetic Energy (T)

$$T = \underbrace{\tfrac{1}{2} m v_{cm}^2}_{\text{KE of CM}} + \underbrace{\sum_i \tfrac{1}{2} m_i v_i'^2}_{\text{KE about CM}}$$

Impulsive Forces - In general, momentum and transferred to a system angular momentum are slowly and the evolution of the system during transfer and the details of the forces are important.

An impulsive force transfers $\vec{P}$ and $\vec{L}$ so quickly that the details of the force are unimportant and the system cannot evolve during the time the force is applied.

Impulse - The total momentum transfer to the system. ~~If the force acts~~

$$\Delta \vec{P} = \int \dot{\vec{P}} \, dt = \int \sum_i F \, dt$$

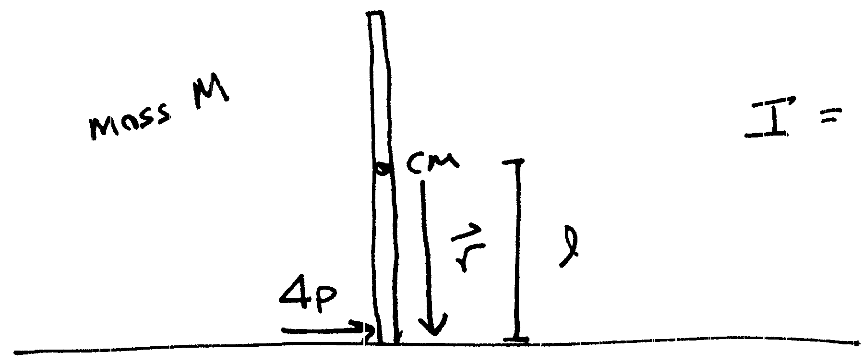
$$= M \int \dot{V}_{cm} \, dt = M \Delta V_{cm}$$

Angular Impulse The total angular momentum transferred

$$\Delta \vec{L} = \int \vec{N} \, dt$$

Note, can either go into rotating system about CM or about origin.

EX Rod of length 2ℓ stands vertically on table. Rod is struck at its base with impulsive force that imparts $\Delta p = \int F\,dt$ momentum to the system



$I =$

Select CM as origin of system, so

$$\vec{L} = \sum_i \vec{r}_i \times m_i \vec{v}_i$$

only rotation about center of mass.

$$\Delta P = m\, v_{cm}$$

$$|\Delta L| = |\vec{r} \times \Delta \vec{P}| = \ell \Delta P = I \Delta \omega$$

$$\Delta \omega = \frac{\ell \Delta P}{I}$$

Moment of Inertia

$$I = \frac{m d^2}{12} = \frac{m(2\ell)^2}{12} = \frac{m\ell^2}{3}$$

$$\Delta\omega = \omega = \frac{l \Delta P}{m l^2/3} = \frac{3 \Delta P}{m l}$$

Velocity of tip immediate after strike

$$V_{tip} = V_{cm} + l\omega$$

$$= \frac{\Delta P}{m} + \frac{3\Delta P}{m} = \frac{4\Delta P}{m}$$

Instantaneous axis of rotation = point on rod with zero velocity.

$$0 = V_{cm} + x\omega$$

$$x = -\frac{V_{cm}}{\omega} = -\frac{1}{3} l$$

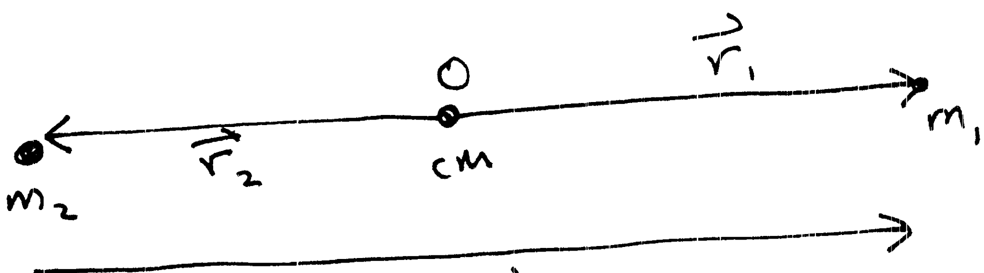
The rod appears to rotate about a point $-\frac{1}{3} l$ (upward) from the center.

Reduced Mass

Consider two masses m_1 and m_2 that interact through a central force $f(r)$.

Let CM be origin, $\dfrac{\sum m_i \vec{r}_i}{M} = 0$

$$m_1\vec{r}_1 + m_2\vec{r}_2 = 0 \implies \vec{r}_2 = -\frac{m_1}{m_2}\vec{r}_1$$



$$\vec{R} = \vec{r}_{21}$$

Displacement from 2 to 1.

Forces

$$\vec{F}_{12} = -f(r_{12})\,\hat{r}_{12}$$

$$\vec{F}_{21} = f(r_{21})\,\hat{r}_{21}$$

<u>EOM</u> mass 1

$$\vec{F}_1 = \vec{F}_{21} = f(r_{21})\,\hat{r}_{21} = \frac{f(r_{21})}{r_{21}}\,\vec{r}_{21}$$

$$= m_1 \frac{d^2\vec{r}_1}{dt^2} \qquad \underline{\text{Newton II}}$$

$$\vec{r}_{21} = \vec{r}_1 - \vec{r}_2 = \vec{r}_1 - \left(-\frac{m_1}{m_2}\vec{r}_1\right)$$

$$= \vec{r}_1\left(1 + \frac{m_1}{m_2}\right) = \vec{r}_1\left(\frac{m_1+m_2}{m_2}\right)$$

$$\vec{r}_1 = \frac{m_2}{m_1+m_2}\,\vec{r}_{21}$$

$$\frac{m_1 m_2}{m_1+m_2}\,\frac{d^2\vec{r}_{21}}{dt^2} = \frac{f(r_{21})}{r_{21}}\,\vec{r}_{21}$$

$\Rightarrow$ Single body dynamics involve separation between the objects $\vec{r}_{21}$ and a modified mass.

Reduced Mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

EOM - Two bodies under a control force

$$\mu \frac{d^2\vec{r}}{dt^2} = f(\vec{r})\,\hat{r}$$

distance betwen objects.

If gravity,

$$\mu \frac{d^2\vec{r}}{dt^2} = \frac{-mMG}{r^2}\,\hat{r}$$

Note, reduced mass replaces inertial mass
NOT gravitational mass.

Ex Returning to our calculation of period for planets. We canceled two m's. Return to the place before cancelation.

$$\tau = 2\pi \sqrt{\frac{m}{k}}\, a^{3/2}$$

This m is the inertial mass $m \to \mu$. k contains gravitational mass.

$$\tau = 2\pi \sqrt{\frac{\mu}{k}}\, a^{3/2}$$

$$= 2\pi \sqrt{\frac{\mu}{m M_\odot G}}\, a^{3/2}$$

$$\mu = \frac{m M_\odot}{m + M_\odot}$$

$$\tau = 2\pi \left((m + M_\odot) G \right)^{-1/2} a^{3/2}$$