Mechanics Fall 2009 - Test 2

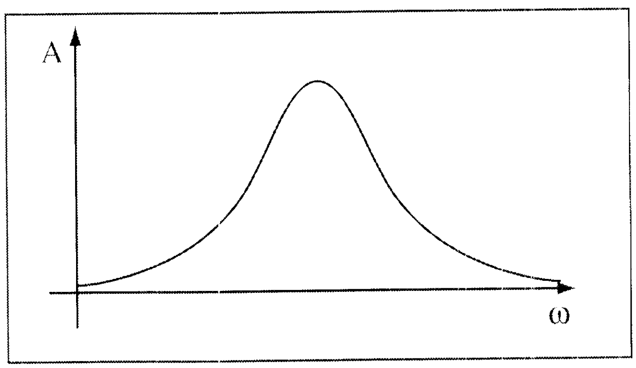
Work four of the five problems. Place the problems in the order you wish them graded. The first two problems form the first half test; the second two problems form the second half test.

Problem 2.1 A cyclotron accelerates particles in circular orbits. Model the force exerted on a mass m in plane polar coordinates by

$$\vec{F} = f(r)\hat{r} + \gamma\hat{\theta}$$

where $f(r)$ is some function of r and γ is a constant. Write the equations of motion in plane polar coordinates. After you have written the equations of motion, solve the $\hat{\theta}$ equation under the assumption that the orbit is circular with $r = a$, where a is a constant with initial condition $\theta(0) = 0$, $\dot{\theta}(0) = 0$. What must $f(r)$ be under this assumption?

Problem 2.2 The figure below shows an experimental measurement of the amplitude A versus the angular frequency ω for a particle of mass m on a spring with spring constant k sliding horizontally on a frictionless surface through a medium that provides a linear drag force $-cv$. The mass and spring constant are known through a separate experiment. Describe how you would use this measurement to determine the drag coefficient c. Define any variables you read from the graph and mark their location on the graph.



Problem 2.3 A bullet travels horizontally through a viscous medium that exerts a quadratic drag force that weakens with distance travelled. If the bullet is moving in the x direction, the drag force is

$$F = -\gamma e^{-x/a} v^2$$

where γ and a are constants. Compute the velocity as a function of position if the initial velocity at the origin is v_0. Does the particle have a maximum range? If yes compute it, if no compute the limiting velocity.

Problem 2.4 Consider the force

$$\vec{F} = -\gamma(x^2\hat{x} + y^2\hat{y})$$

where γ is a constant. Is this force conservative, justify? If a particle of mass m is released at the point $(a, a, 0)$ and travels under this force toward the origin, how fast is the particle moving at the origin?

Problem 2.5 A two-dimensional isotropic harmonic oscillator has potential function $V = \frac{1}{2}k(x^2 + y^2)$ for a particle of mass m. Solve for the trajectory of the particle if the initial conditions as $x(0) = a$, $y(0) = 0$, $\dot{x}(0) = 0$, and $\dot{y}(0) = v_0$. Report the trajectory as $x(t)$ and $y(t)$.

2.1 $\vec{F} = f(r)\hat{r} + \gamma\hat{\theta}$

From kinematics sheet, in plane polar coordinates

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

<u>Newton II</u> $\vec{F} = m\vec{a}$

<u>EOM</u> $f(r) = m\ddot{r} - \cancel{mr}\dot{\theta}^2 = m\ddot{r} - mr\dot{\theta}^2$

$\gamma = mr\ddot{\theta} + 2\dot{r}\dot{\theta}m$

+10

Assume $r = a$, EOM become

Note, a is a constant not acceleration.

+5

$f(r) = -ma\dot{\theta}^2$

$\gamma = ma\ddot{\theta}$

<u>Integrate twice</u>

$$\theta(t) = \theta_0 + \dot{\theta}_0 t + \frac{1}{2}\frac{\gamma}{ma}\cancel{\theta}^2$$

<u>Apply initial conditions</u>

$$\theta(t) = \frac{1}{2}\frac{\gamma}{ma}t^2$$

+5

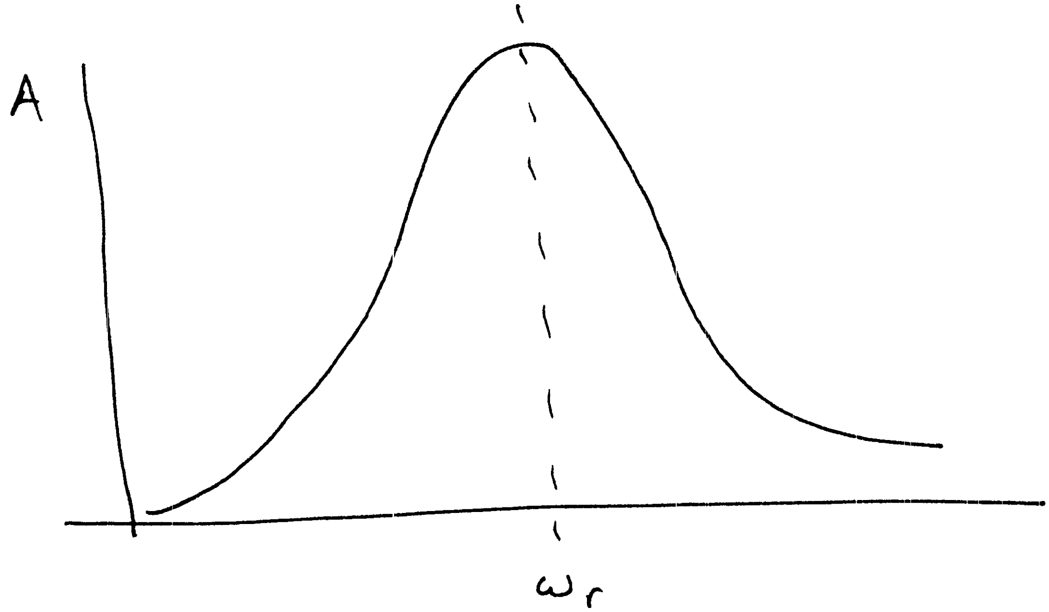
1.2

The force is the

$$f(r) = -ma\left(\frac{\gamma t}{ma}\right)^2$$

$$= -\frac{\gamma^2 t^2}{ma} + 5$$

2.2



Read resonant frequency off graph.

$$\omega_r^2 = \omega_0^2 - 2\gamma^2$$

$$\gamma^2 = \frac{\omega_0^2 - \omega_r^2}{2} = \frac{k/m - \omega_r^2}{2}$$

$$\gamma = \frac{c}{2m} \qquad c = 2m\gamma$$

$$c = 2m\left(\frac{k/m - \omega_r^2}{2}\right)^{1/2}$$

(2.3)

$$F = ma = m\frac{dv}{dt} = mv\frac{dv}{dx}$$

$$= -\gamma e^{-x/a} v^2$$

+12

$$\int_{v_0}^{v} \frac{dv}{v} = \frac{-\gamma}{m}\int_0^{x} e^{-x/a}\, dx$$

+5

$$\ln(v/v_0) = \frac{\gamma a}{m} e^{-x/a}\Big|_0^x$$

$$= -\frac{\gamma a}{m}\left(1 - e^{-x/a}\right)$$

$$v(x) = v_0 e^{-\gamma a/m(1-e^{-x/a})}$$

+8

Limiting $v = v_0 e^{-\gamma a/m}$ +3

2.4

$$\nabla \times F = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\gamma x^2 & -\gamma y^2 & 0 \end{vmatrix} = 0$$ +10

force is conservative.

Potential Function $\vec{F} = -\nabla V$

$$V = \frac{\gamma x^3}{3} + \frac{\gamma y^3}{3}$$ (Let origin be zero of V) +5

If released at ~~origin~~, $(a, a, 0)$ the initial energy is

$$U_{sys} = V(a, a, 0) = \frac{2\gamma a^3}{3}$$ +5

At origin $V = 0$ by our choice of constant, so all energy is kinetic

$$U_{sys} = \frac{2\gamma a^3}{3} = \frac{1}{2} m v^2$$

$$v = \sqrt{\frac{4}{3} \frac{\gamma a^3}{m}}$$ +5

2.5 $\quad V = \frac{1}{2}k(x^2+y^2)$

$$\vec{F} = -\nabla V = -kx\hat{x} - ky\hat{y} \quad +5$$

EOM

$$\omega_0^2 = \frac{k}{m}$$

$$\left.\begin{aligned} m\ddot{x} &= -kx \\ m\ddot{y} &= -ky \end{aligned}\right] +5$$

Solutions

$$\left.\begin{aligned} y(t) &= C\sin\omega_0 t + D\cos\omega_0 t \\ x(t) &= A\sin\omega_0 t + B\cos\omega_0 t \end{aligned}\right] +5$$

x-initial condition

$$\left.\begin{aligned} x(0) &= B = a \\ \dot{x}(0) &= \omega_0 A = 0 \quad \Rightarrow A = 0 \end{aligned}\right] +5$$

$$x(t) = a\cos\omega_0 t$$

y-initial condition

$$\left.\begin{aligned} y(0) &= 0 = D \\ \dot{y}(0) &= C\omega_0 = v_0 \quad C = v_0/\omega_0 \end{aligned}\right] +5$$

$$y(t) = \frac{v_0}{\omega_0}\sin\omega_0 t$$