Quantum Mechanics Summer 2003- Final Exam

Tuesday, 5:30pm at the end of my office hours.

I have not worked this yet so watch for updates over the weekend. The test will not be returned until all students seeking a grade in the class have turned it in, so early next semester.

Problem 1 Consider a particle of mass m moving in a one dimensional potential $V = \frac{1}{4}kx^4$.

(a) Write the Hamiltonian of the system.

(b) Calculate the time rate of change of the average position $< X >$ and the average momentum $< P >$.

(c) Write the uncertainty relation for the momentum and energy.

(d) Suppose the system in in state $\psi = A\exp(-ax^2)$, where A is a normalization constant. Find the normalized wave function.

(e) Calculate the expectation value $< \psi|X|\psi >$ and $< \psi|P|\psi >$.

(f) Compute the expectation value $< \psi|X^2|\psi >$ and $< \psi|P^2|\psi >$. You may find the integral

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{\sqrt{\pi}}{2a^{\frac{3}{2}}}$$

useful.

(g) Write the uncertainties σ_X and σ_P in terms of the quantities in (e) and (f). Verify Heisenberg's Uncertainty Relation for this state.

Problem 2 Consider a 3 state system spanned by the vectors $\{|1>, |2>, |3>\}$. The hamiltonian of the system (in this basis is),

$$\hat{H} = \hbar\omega \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

The system is in state $|\psi(0)> = \frac{1}{2}|1> + \frac{1}{\sqrt{2}}|2> + \frac{1}{2}|3>$ at time $t = 0$.

(a) Is the Hamiltonian Hermitian? (Ok so I asked all right).

(b) If the energy of the system is measured what are the possible outcomes for this state vector and with probabilities?

(c) Calculate $|\psi(t)>$.

Now consider the observables.

$$\hat{A} = a \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\hat{B} = b \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(d) Which of the following sets of operators are complete sets of commuting observables? $\{H\}$, $\{A\}$, $\{A, B\}$, $\{A, H\}$, or $\{H, B\}$. Justify.

(e) Calculate the expectation value of B as a function of time.

(f) For $|\psi(0)>$, what outcomes of a measurement of B are possible and with what probabilities?

Problem 3 The lowest three orthonormal energy eigenstates of the the simple harmonic oscillator are

$$\phi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\eta^2/2}$$

$$\phi_1 = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \sqrt{2}\eta e^{-\eta^2/2}$$

$$\phi_2 = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2}}(2\eta^2 - 1)e^{-\eta^2/2}$$

where

$$\eta = \left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}} x$$

The energy of a simple harmonic oscillator is

$$E_n = (n + \frac{1}{2})\hbar\omega$$

where $\omega = \sqrt{\frac{k}{m}}$.

(a) Verify that ϕ_0 is an eigenstate of the the Hamiltonian and find its eigenvalue by allowing the Hamiltonian to operate on the state.

(b) Construct ϕ_3 by applying the appropriate ladder operator. Do not bother with renormalization.

The system is in the state

$$\psi(x, t = 0) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\eta^2/2} \frac{1}{\sqrt{2}} \left[1 + \sqrt{2}\eta\right]$$

(c) What energy values could be observed and with what probabilities? Hint you do not need to integrate to do this. Hint 2, write the ψ as an expansion in terms of the functions above.

(d) Calculate $\psi(x, t)$

Problem 4 Consider an isotopic harmonic oscillator with potential

$$V(x, y, z) = \frac{1}{2}k(x^2 + y^2 + z^2) = \frac{1}{2}kr^2$$

(a) Write the Schrodinger equation. Separate the Schrodinger equation into a radial and angular parts. Solve the angular part.

(b) Write the differential equation the radial part satisfies. Write both the $R(r)$ and $u(r)$ equations.

(c) Solve the radial equation for $\ell = 0$ and compute the ground state energy. Hint, compare the radial equation to the equation for a simple one dimensional harmonic oscillator. Hint 2, there is a trick here where one of the solutions $R(r) = u(r)/r$ is singular and must be discarded.

(d) Suppose the wave function of the system is $\psi = NR(r)\cos\theta$, where N is a normalization and $R(r)$ is the solution to the radial equation. What values could be measured for L^2 and what values for L_z.

Bonus Problem Jon Hubbard's dad has a dump truck. Its mass is about 5000kg. Its difficult to do quantum mechanics with an object this size, so let's assume $\hbar$ is a little larger. Let $\hbar = 50000$Js. Jon makes the mistake of driving the dump truck into a canyon (modelled as a one dimensional infinite square well) with width 10m. What is the minimum speed of the dump truck toward the canyon wall?