Quantum Mechanics Fall 2003- Homework Set 1

Postulates and Mathematics

Due 5:00pm September 5, 2003 in physics office or under my door.

Reading

We are going to wallow in the first 200 pages of Griffiths, so you might as well read it all. For this homework set, the following are useful.

Section 1.3

Chapter 3

Griffith's Problems

1.2

3.1

3.2

3.6

3.24 (Parts a and b)

3.28

Additional Problems

Problem A1 Prove the following properties of the commutator of operators $\hat{A}$, $\hat{B}$, and $\hat{C}$.

(a) $[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$

(b) $[\hat{A}, [\hat{B}, \hat{C}]] + [\hat{B}, [\hat{C}, \hat{A}]] + [\hat{C}, [\hat{A}, \hat{B}]] = 0$

(c) $[\hat{A}, \hat{B}]^\dagger = [\hat{B}^\dagger, \hat{A}^\dagger]$

Problem A2 Consider the position operator $\hat{X}$ and the momentum operator $\hat{P}$ that obey the canonical commutation relation $[\hat{X}, \hat{P}] = i\hbar$.

(a) Show $[\hat{X}, \hat{P}^2] = 2i\hbar P$

(b) Arguing by induction, you can show the first previous result generalizes to $[\hat{X}, \hat{P}^n] = i\hbar n P^{n-1}$. Use this result and the Taylor expansion to show.

$$[\hat{X}, F(\hat{P})] = i\hbar \frac{dF(P)}{dP}$$

where $F(P)$ is some function of P. It is also true but you do not have to show it that

$$[\hat{P}, G(\hat{X})] = -i\hbar \frac{dG(X)}{dX}$$

Write this one on the front cover of your notes, it is as useful as they come.

People seem to be having trouble with this. A quick Taylor expansion review: The Taylor expansion of a function about zero is:

$$f(x) = \sum_n \left. \frac{d^n f}{dx^n} \right|_0 \frac{x^n}{n!}$$

This expansion can be differentiate term by term giving

$$\frac{df(x)}{dx} = \sum_n n \left. \frac{d^n f}{dx^n} \right|_0 \frac{x^{n-1}}{n!}$$

Problem A3 Prove that the product of two unitary operators is unitary.

Problem A4 Find the triple angle trig formulas by examining

$$e^{3i\theta} = (e^{i\theta})^3$$

Cohen-Tannoudji Problems - in H_{II}

Problem CT 2.1 Let $|\phi_n>$ be orthonormal eigenstates of the operator $\hat{H}$ such that $\hat{H}|\phi_n >= E_n|\phi_n >$. Assume $|\phi_n>$ form an othonormal basis. Consider the operator $U(m,n) = |\phi_m >< \phi_n|$.

(a) Calculate the adjoint of $U(m,n)$.

(b) Calculate the commutator $[\hat{H}, U(m,n)]$

(c) Prove the relation:

$$U(m,n)U^\dagger(p,q) = \delta_{n,q} U(m,p)$$

(d) Calculate

$$\mathrm{Tr}\{U(m,n)\}$$

The trace of an operator, $\hat{A}$, is defined as

$$\mathrm{Tr}(\hat{A}) = \sum_i < \phi_i|\hat{A}|\phi_i >$$

Problem CT 2.8 Consider the Hamiltonian of a one dimensional particle

$$\hat{H} = \frac{\hat{P}^2}{2m} + V(\hat{X})$$

where $V(x)$ is the potential energy. The eigenvalues of the Hamiltonian are

$$\hat{H}|\phi_n >= E_n|\phi_n >$$

The operators satisfy $[\hat{X}, \hat{P}] = i\hbar$. Show that

$$< \phi_n|\hat{P}|\phi_{n'} >= \alpha < \phi_n|\hat{X}|\phi_{n'} >$$

where α is a constant you determine.

Big Hint: Examine

$$< \phi_n|[\hat{H}, \hat{X}]|\phi_{n'} >$$

Big Hint 2: Take the adjoint of the eigenvalue equation to yield

$$< \phi_n|\hat{H}^\dagger = E_n^* < \phi_n|$$

which since a physical observable is Hermitian and the energy is real is

$$< \phi_n|\hat{H} = E_n < \phi_n|$$

This shows how the Hamiltonian acts to the left.

Problem CT 2.9 Consider the Hamiltonian of a physical system $\hat{H}$ The eigenvalues of the Hamiltonian are

$$\hat{H}|\phi_n >= E_n|\phi_n >$$

where E_n is the energy. The operators satisfy $[\hat{X}, \hat{P}] = i\hbar$.

(a) Show for an arbitrary operator $\hat{A}$ that $< \phi_n|[\hat{H}, \hat{A}]|\phi_n >= 0$

Now consider the one dimensional Hamiltonian

$$\hat{H} = \frac{\hat{P}^2}{2m} + V(\hat{X})$$

where $V(x)$ is the potential energy. The operators satisfy $[\hat{X}, \hat{P}] = i\hbar$.

(b) Compute the commutators $[\hat{H}, \hat{X}]$, $[\hat{H}, \hat{P}]$, and $[\hat{H}, \hat{X}\hat{P}]$.

Big Hint. For the problems which follow, use the commutators found in part (b) in the general relation in part (a)

(c) Show the matrix element $< \phi_n|\hat{P}|\phi_n >= 0$. This matrix element is the average momentum.

(d) Establish a relation between the average kinetic energy

$$< T >=< \phi_n|\frac{\hat{P}^2}{2m}|\phi_n >$$

and the average work done by the force represented by the potential

$$< W >= - < \phi_n|\hat{X}\frac{dV(\hat{X})}{dX}|\phi_n >$$