Addition of Angular Momentum

Consider a system formed of two particles 1 and 2 with single particle states $|\psi_1\rangle$ and $|\psi_2\rangle$. The combined state of the system can be represented as

$$|\psi\rangle \equiv |\psi_1\rangle \otimes |\psi_2\rangle$$

called a <u>tensor product</u>. It simply means we have to provide the state for particle 1 and particle 2.

We can extend single particle operators to this new state. Let $\vec{S}_1$ be the operator associated with the spin of particle 1 and $\vec{S}_2$ with particle 2.

Classically, the total angular momentum of the two particle system is

$$\vec{S} = \vec{S}_1 + \vec{S}_2$$

$$\vec{S}_1 = (S_{1x}, S_{1y}, S_{1z})$$

$$\vec{S}_2 = (S_{2x}, S_{2y}, S_{2z})$$

$$\vec{S} = \vec{S}_1 + \vec{S}_2 = (\underbrace{S_{1x} + S_{2x}}_{= S_x}, \underbrace{S_{1y} + S_{2y}}_{= S_y}, \underbrace{S_{1z} + S_{2z}}_{= S_z})$$

But now $\vec{S}$ acts on a two particle state.

Define $\hat{S}_{1x} |\psi\rangle = \hat{S}_{1x} \left(|\psi_1\rangle \otimes |\psi_2\rangle \right)$

$$\equiv \left(\hat{S}_{1x} |\psi_1\rangle \right) \otimes |\psi_2\rangle$$

and

$$\hat{S}_{2x} |\psi\rangle \equiv |\psi_1\rangle \otimes \left(S_{2x} |\psi_2\rangle \right)$$

Similar definitions allow us to extend all components of $\vec{S}$ to the two particle state.

The operator $\vec{S}$ is a generalized angular momentum which is defined by the commutation relations

$$[S_x, S_y] = i\hbar S_z \qquad [S_y, S_z] = i\hbar S_x$$
$$[S_z, S_x] = i\hbar S_y$$

Note, operators for different particles commute.

$$[\vec{S}_{1i}, \vec{S}_{2j}] = 0$$

so

$$[S_x, S_y] = [S_{1x} + S_{2x}, S_{1y} + S_{2y}]$$

$$= [S_{1x}, S_{1y}] + \underset{\overset{\shortparallel}{0}}{[S_{1x}, S_{2y}]} + \underset{\overset{\shortparallel}{0}}{[S_{2x}, S_{1y}]}$$
$$+ [S_{2x}, S_{2y}]$$

$$= i\hbar S_{1z} + i\hbar S_{2z} = i\hbar S_z$$

The other commutators follow through similar methods.

Since $\vec{S}$ satisfies the commutation relations defining angular momentum, it has all the properties of an angular momentum.

$$\Rightarrow \quad S^2|sm\rangle = s(s+1)\hbar^2|sm\rangle$$

$$S_z|sm\rangle = m\hbar|sm\rangle$$

$$S_+|sm\rangle = \hbar\sqrt{s(s+1)-m(m+1)}\,|s\ m+1\rangle$$

$$S_-|sm\rangle = \hbar\sqrt{s(s+1)-m(m-1)}\,|s\ m-1\rangle$$

$$S_+ = S_x + iS_y = (S_{1x}+S_{2x}) + i(S_{1y}+S_{2y})$$

$$= S_{1+} + S_{2+}$$

$$S_- = S_x - iS_y = S_{1-} + S_{2-}$$

Since we have lowering operators, we could construct all states if we could find $|ss\rangle$.

Let's be a little more concrete, consider two spin 1/2 particles. The basis states for one of the particles is $\{|+\rangle, |-\rangle\}$ so a basis for the two particle system is $\{|+\rangle|+\rangle, |+\rangle|-\rangle, |-\rangle|+\rangle, |-\rangle|-\rangle\}$

where $|+\rangle|+\rangle \equiv |+\rangle \otimes |+\rangle$

We are looking for the eigenstates of S^2, S_z which could be any linear combination of those four vectors.

If the universe made sense, the highest angular momentum state would be $|+\rangle|+\rangle$. Let's see if it is an eigenstate.

$$
\begin{aligned}
S_z|+\rangle|+\rangle &= (S_{1z}+S_{2z})|+\rangle|+\rangle \\
&= S_{1z}|+\rangle|+\rangle + S_{2z}|+\rangle|+\rangle \\
&= (S_{1z}|+\rangle)|+\rangle + |+\rangle(S_{2z}|+\rangle) \\
&= \frac{\hbar}{2}|+\rangle|+\rangle + \frac{\hbar}{2}|+\rangle|+\rangle = \hbar|+\rangle|+\rangle
\end{aligned}
$$

So $|+\rangle|+\rangle$ is an eigenstate of S_z with eigenvalue $\hbar \Rightarrow m=1$.

<u>Try S^2</u>

$$S^2 = (\vec{S}_1 + \vec{S}_2)\cdot(\vec{S}_1 + \vec{S}_2)$$

$$= S_1^2 + S_2^2 + 2\vec{S}_1\cdot\vec{S}_2$$

$$\vec{S}_1\cdot\vec{S}_2 = (S_{1x}S_{2x} + S_{1y}S_{2y} + S_{1z}S_{2z})$$

Write in terms of raising and lowering ops

$$\cancel{S_{1x} = \tfrac{1}{2}S(}$$

$$S_{1x} = \frac{1}{2}(S_{1+} + S_{1-})$$

$$S_{1y} = \frac{1}{2i}(S_{1+} - S_{1-})$$

$$S_{1x}S_{2x}|+\rangle|+\rangle = \frac{1}{4}(S_{1+} + S_{1-})(S_{2+} + S_{2-})|+\rangle|+\rangle$$

$$= \frac{1}{4}(S_{1+} + S_{1-})\left(|+\rangle(S_{2-}|+\rangle)\right)$$

$$= \frac{\hbar}{4}(S_{1+} + S_{1-})|+\rangle|-\rangle = \frac{\hbar^2}{4}|-\rangle|-\rangle$$

Normalized Lowering Op

$$S_- |+\rangle = \hbar \sqrt{\underbrace{\tfrac{1}{2}\left(\tfrac{1}{2}+1\right)}_{3/4} - \underbrace{\tfrac{1}{2}\left(\tfrac{1}{2}-1\right)}_{1/4}}\; |-\rangle$$

$$= \hbar |-\rangle$$

$$S_{1y} S_{2y} |+\rangle|+\rangle = -\frac{1}{4} \left(S_{1+} - S_{1-}\right)\left(S_{2+} - S_{2-}\right) |+\rangle|+\rangle$$

$$= +\frac{\hbar}{4} \left(S_{1+} - S_{1-}\right) |+\rangle|-\rangle$$

$$= -\frac{\hbar^2}{4} |-\rangle|-\rangle$$

$$S_{1z} S_{2z} |+\rangle|+\rangle = \frac{\hbar^2}{4} |+\rangle|+\rangle$$

$$\cancel{\vec{S}_1 \cdot \vec{S}_2 |+\rangle = S_{1x} S_{2x} + S_{1y}}$$

$$\vec{S}_1 \cdot \vec{S}_2 |+\rangle|+\rangle = \frac{\hbar^2}{4} |-\rangle|-\rangle - \frac{\hbar^2}{4} |-\rangle|-\rangle$$

$$+ \frac{\hbar^2}{4} |+\rangle|+\rangle$$

$$= \frac{\hbar^2}{4} |+\rangle|+\rangle$$

$$S^2|+\rangle|+\rangle = \left(S_1^2 + S_2^2 + 2\vec{S}_1\cdot\vec{S}_2\right)|+\rangle|+\rangle$$

$$= \left(\frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 + \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2 + 2\cdot\frac{\hbar^2}{4}\right)|+\rangle|+\rangle$$

$$= 2\hbar^2|+\rangle|+\rangle$$

$$= 1(1+1)\hbar^2|+\rangle|+\rangle$$

$|+\rangle|+\rangle$ is an eigenstate of S^2 with eigenvalue

$2\hbar^2 \Rightarrow s = 1$

$$|+\rangle|+\rangle = |1\,1\rangle = |s\,m\rangle$$

We've got the highest s state, now use S_- to construct the other states.

$$S_-|11\rangle = \hbar\sqrt{1(1+1)-1(1-1)}\;|10\rangle$$

$$= \sqrt{2}\,\hbar\,|10\rangle$$

$$S_-|11\rangle = (S_{1-} + S_{2-})\,|+\rangle|+\rangle$$

$$= \hbar|-\rangle|+\rangle + \hbar|+\rangle|-\rangle$$

$$\Rightarrow \boxed{|10\rangle = \frac{1}{\sqrt{2}}|+\rangle|-\rangle + \frac{1}{\sqrt{2}}|-\rangle|+\rangle}$$

<u>Keep going</u>

$$S_-|10\rangle = \hbar\sqrt{1(1+1)-0(0-1)}\;|1\,-1\rangle$$

$$= \sqrt{2}\,\hbar\,|1\,-1\rangle$$

$$= (S_{1-} + S_{2-})\,\frac{1}{\sqrt{2}}\left(|+\rangle|-\rangle + |-\rangle|+\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(\hbar|-\rangle|-\rangle + \hbar|-\rangle|-\rangle\right.$$

$$= \sqrt{2}\,\hbar\,|-\rangle|-\rangle$$

So $|1\ {-1}\rangle = |-\rangle|-\rangle$ as expected

So for $s=1$ we have three states called <u>triplet</u> states

$$|1\,1\rangle = |+\rangle|+\rangle$$

$$|1\,0\rangle = \frac{1}{2}\left(|+\rangle|-\rangle + |-\rangle|+\rangle\right)$$

$$|1\ {-1}\rangle = |-\rangle|-\rangle$$

But that's not enough, the dimension of our state space is 4 and we have only 3 basis vectors.

The fourth orthogonal vector is easy to see

$$\frac{1}{\sqrt{2}}\left(|+\rangle|-\rangle - |-\rangle|+\rangle\right)$$

This is called the singlet state and has $s=0$ and $m=0$, which you should test since it looks like a great test question.

Singlet $\quad |00\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle|-\rangle - |-\rangle|+\rangle\right)$

What can we measure?

I. Total angular momentum or z-component
S^2, S_z

II Individual particle angular momentum and z-component
$S_1^2, S_2^2, S_{1z}, S_{2z}$

Can we measure both at the same time?

Ex A system is prepared in the $|10\rangle$ state. What is the probability the first particle is spin up?

Sln

$$|10\rangle = \underbrace{\frac{1}{\sqrt{2}}}_{C_{1+}}|+\rangle|-\rangle + \underbrace{\frac{1}{\sqrt{2}}}_{C_{1-}}|-\rangle|+\rangle$$

$$P(\text{particle 1 spin up}) = C_{1+}^* C_{1+} = \frac{1}{2}$$

Suppose the first particle is measured ~~the~~ to have spin up for this state. What total angular momentum could be observed with what probability?

The measurement of the first particle as spin up projects the system into the state

$$|\psi\rangle = |+\rangle|-\rangle$$

Write this state in terms of the eigenstates of total angular momentum.

$$|+\rangle|-\rangle = \frac{1}{\sqrt{2}}|10\rangle + \frac{1}{\sqrt{2}}|00\rangle$$

$$P(s=1) = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2} \qquad S^2 = 2\hbar^2$$

$$P(s=0) = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2} \qquad S^2 = 0$$

Suppose this hadn't worked. We could always write the matrix representing S^2 in the $\{|+\rangle|+\rangle, |+\rangle|-\rangle, |-\rangle|+\rangle, |-\rangle|-\rangle\}$ bases and find its mutual eigenvectors with S_z.