Angular Momentum Matrices

The generalized angular momentum matrices J^2, J_z had quantum numbers

$$j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2 \cdots$$

The integer j's have position space eigenfunctions and can be used to represent orbital angular momentum. The $\tfrac{1}{2}$ integer j's do not have eigenfunctions, but do represent a physical property, the intrinsic angular momentum or spin. We must use matrices to investigate $j = \tfrac{1}{2}, \tfrac{3}{2} \cdots$

Build Angular Momentum Matrices

$\underline{j=0} \quad \Longrightarrow \quad m = 0$

Basis States $\{|00\rangle\}$

J^2 and J_z are 1×1 matrices

$$J^2 = \left(\langle 00|J^2|00\rangle \right) = \left(0 \right)$$

$$J_z = \left(\langle 00|J_z|00\rangle \right) = \left(0 \right)$$

$j = 1/2$, $m = \pm 1/2$

Basis States $\left\{ |\tfrac{1}{2}\,\tfrac{1}{2}\rangle, |\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle \right\}$

$$|+\rangle \equiv |\tfrac{1}{2}\,\tfrac{1}{2}\rangle$$

$$|-\rangle \equiv |\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle$$

$$J^2|+\rangle = \tfrac{1}{2}(\tfrac{1}{2}+1)\hbar^2|+\rangle = J^2|-\rangle$$

$$= \tfrac{3}{4}\hbar^2|+\rangle$$

$$J_z|+\rangle = \tfrac{1}{2}\hbar|+\rangle \qquad J_z|-\rangle = -\tfrac{1}{2}\hbar|-\rangle$$

A vector in the $\{|+\rangle, |-\rangle\}$ basis is called a "spinor".

We will use the operators $\vec{S}$, S^2, S_z for spin $1/2$.

$$S^2 = J^2 \text{ etc.}$$

Find S^2, S_x, S_y, S_z

$$S^2 = \begin{pmatrix} \langle +|S^2|+\rangle & \langle +|S^2|-\rangle \\ \langle -|S^2|+\rangle & \langle -|S^2|-\rangle \end{pmatrix}$$

$$= \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S_z = \begin{pmatrix} \langle +|S_z|+\rangle & \langle +|S_z|-\rangle \\ \langle -|S_z|+\rangle & \langle -|S_z|-\rangle \end{pmatrix}$$

$$= \frac{1}{2}\hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

To calculate S_x, S_y work on raising/lowering ops

$$\hat{S}_+ = \hat{S}_x + i\hat{S}_y \qquad \hat{S}_- = \hat{S}_x - i\hat{S}_y$$

$$S_x = \frac{1}{2}(\hat{S}_+ + \hat{S}_-) \qquad S_y = \frac{1}{2i}(\hat{S}_+ - \hat{S}_-)$$

Normalized Operators

$$\hbar\sqrt{j(j+1) - m(m+1)}\, |j\ m+1\rangle = J_+ |jm\rangle$$

$$\hbar\sqrt{j(j+1) - m(m-1)}\, |j\ m-1\rangle = J_- |jm\rangle$$

So

$$S_+|+\rangle = S_+|\tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

$$= 0$$

$$S_+|-\rangle = S_+|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$$

$$= \hbar\sqrt{\underbrace{\tfrac{1}{2}(\tfrac{1}{2}+1)}_{3/4} - \underbrace{(-\tfrac{1}{2})(-\tfrac{1}{2}+1)}_{+\frac{1}{4}}}\ |\tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

$$S_+|-\rangle = \hbar|+\rangle$$

$$\langle+|S_+|-\rangle = \hbar\langle+|+\rangle = \hbar$$

$$\langle-|S_-|+\rangle = \hbar$$

$$S_x = \begin{pmatrix} \langle+|\frac{1}{2}(S_++S_-)|+\rangle & \langle+|\frac{1}{2}(S_++S_-)|-\rangle \\ \langle-|\frac{1}{2}(S_++S_-)|+\rangle & \langle-|\frac{1}{2}(S_++S_-)|-\rangle \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{1}{2}\langle+|S_+|-\rangle \\ \frac{1}{2}\langle-|S_-|+\rangle & 0 \end{pmatrix}$$

$$= \frac{1}{2}\hbar\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

<u>Likewise</u>

$$S_y = \frac{1}{2}\hbar\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Pauli Spin Matrices

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{S}_x = \frac{1}{2}\hbar\,\hat{\sigma}_x \qquad \hat{S}_y = \frac{1}{2}\hbar\,\hat{\sigma}_y \qquad \hat{S}_z = \frac{1}{2}\hbar\,\hat{\sigma}_z$$

Let's try L_x, L_y, L_z for $l=1$.

$$L^2 = l(l+1)\hbar^2 = 2\hbar^2$$

$$L^2 = 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

State space $|1\rangle \equiv |1\,1\rangle$ $|0\rangle \equiv |1\,0\rangle$ $|-1\rangle = |1\,-1\rangle$

$$L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

As before $\hat{L}_x = \frac{1}{2}(\hat{L}_+ + \hat{L}_-)$

$$L_x = \begin{pmatrix} \langle 1|L_x|1\rangle & \langle 1|L_x|0\rangle & \langle 1|L_x|-1\rangle \\ \langle 0|L_x|1\rangle & \langle 0|L_x|0\rangle & \langle 0|L_x|-1\rangle \\ \langle -1|L_x|1\rangle & \langle -1|L_x|0\rangle & \langle -1|L_x|-1\rangle \end{pmatrix}$$

Only non-zero term are elements where bra and ket 1 apart.

$$L_+|0\rangle = L_+|10\rangle = \hbar\sqrt{1(1+1) - 0(0+1)}\ |1\rangle$$

$$= \sqrt{2}\hbar\,|1\rangle$$

$$L_+|-1\rangle = \hbar\sqrt{1(1+1) + 1(-1+1)}\ |0\rangle$$

$$= \sqrt{2}\hbar\,|0\rangle$$

$$\langle 1|S_x|0\rangle = \frac{1}{2}\langle 1|S_+ + S_-|0\rangle = \frac{\hbar}{\sqrt{2}}$$

$$\langle 0|S_x|-1\rangle = \frac{1}{2}\langle 0|S_+ + S_-|-1\rangle = \frac{\hbar}{\sqrt{2}}$$

$$L_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

other elements can be found by Hermiticity