Angular Momentum

As we move to 3-d, it would be natural to expect angular momentum to play an important role. Let's investigate by:

1. Write classical quantity
2. Quantize to produce quantum mechanical operator.
3. Find eigenvalues and eigenvectors.
4. Write eigenvectors in position space to produce wave functions.

Recall $[\hat{X}_i, \hat{X}_j] = 0 \quad [\hat{P}_i, \hat{P}_j] = 0$

$$[\hat{X}_i, \hat{P}_j] = i\hbar\,\delta_{ij}$$

Classical Angular Momentum $\vec{L}$

$$\vec{L} = \vec{r} \times \vec{p} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

$$= \hat{x}(y p_z - z p_y) - \hat{y}(x p_z - z p_x) + \hat{z}(x p_y - y p_x)$$

$$= (L_x, L_y, L_z)$$

$$L_x = y p_z - z p_y \xrightarrow{\text{Quantize}} \hat{L}_x = \hat{Y}\hat{P}_z - \hat{Z}\hat{P}_y$$

$$L_y = z p_x - x p_z \Longrightarrow \hat{L}_y = \hat{Z}\hat{P}_x - \hat{X}\hat{P}_z$$

$$L_z = x p_y - y p_x \Longrightarrow \hat{L}_z = \hat{X}\hat{P}_y - \hat{Y}\hat{P}_x$$

Do we need to symmetrize?

$$\hat{L}_x = \frac{1}{2}\left(\hat{Y}\hat{P}_z - \hat{Z}\hat{P}_y + \hat{P}_z\hat{Y} - \hat{P}_y\hat{Z}\right)$$

$\Rightarrow$ No, all products commute in pairs.

Commutation Relations - You may easily prove

$$[\hat{L}_x, \hat{L}_y] = i\hat{L}_z \qquad [\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x$$

$$[\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y$$

$\Rightarrow$ Angular momentum operators do not commute $\Rightarrow$ Different components of angular momentum are not compatible observables $\Rightarrow$ cannot be measured simultaneously.

Dfn Generalized Angular Momentum ($\hat{\vec{J}}$)

Any set of operators that obey

$$[\hat{J}_x, \hat{J}_y] = i\hbar \hat{J}_z \qquad [\hat{J}_y, \hat{J}_z] = i\hbar \hat{J}_x$$

$$[\hat{J}_z, \hat{J}_x] = i\hbar \hat{J}_y$$

Dfn Total Angular Momentum (J^2)

$$J^2 = \vec{J} \cdot \vec{J} = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$$

Commutator

$$[\hat{J}^2, \hat{J}_x] = [\hat{J}^2, \hat{J}_y] = [\hat{J}^2, \hat{J}_z] = 0$$

$\Rightarrow$ The total angular momentum is a compatable observable with any component of angular momentum.

Define Ladder Operators

$$\hat{J}_+ = \hat{J}_x + i\,\hat{J}_y$$

$$\hat{J}_- = \hat{J}_+^{\,\dagger} = \hat{J}_x - i\,\hat{J}_y$$

Naturally, we will need to show these actually act as ladder operators.

Ladder Operator Commutation Relations

$$[\hat{J}_z, \hat{J}_\pm] = \pm\hbar\,\hat{J}_\pm$$

$$[\hat{J}_+, \hat{J}_-] = 2\hbar\,\hat{J}_z$$

$$[\hat{J}^2, \hat{J}_\pm] = 0 \quad \text{(obviously)}$$

Since $[\hat{J}^2, \hat{J}_z] = 0$, the total angular momentum and the z-component of angular momentum are compatable, simultaneously measurable, observable. As such, there is a mutual basis of eigenvectors where

$$\hat{J}^2 |jm\rangle = \underbrace{j(j+1)\hbar^2}_{\text{eigenvalue of } \hat{J}^2} |jm\rangle$$

$$\hat{J}_z |jm\rangle = \underbrace{m\hbar}_{\text{eignvalue of } \hat{J}_z} |jm\rangle$$

$\Rightarrow$ This odd form for the eigenvalue does not at this point place any restrictions on the eigenvalue,

$\Rightarrow$ By the rules of QM, the measurable values of the total angular momentum $|\vec{J}|$ are

$$\cancel{\sqrt{j(j+1)\hbar^2}} \quad \sqrt{j(j+1)\hbar^2}$$

and the z-component of angular momentum

$$m\hbar$$

But what are j, m?

Work on the operators ; Solve for $\hat{J}_x, \hat{J}_y$

$$\hat{J}_x = \frac{1}{2}(\hat{J}_+ + \hat{J}_-)$$

$$\hat{J}_y = \frac{1}{2i}(\hat{J}_+ - \hat{J}_-)$$

Find $\hat{J}_- \hat{J}_+$

$$\hat{J}_-\hat{J}_+ = (\hat{J}_x - i\hat{J}_y)(\hat{J}_x + i\hat{J}_y)$$

I'm bored with hats

$$= J_x^2 + J_y^2 + i[J_x, J_y]$$

$$= J_x^2 + J_y^2 - \hbar J_z$$

$$= J^2 - J_z^2 - \hbar J_z$$

Likewise

$$J_+ J_- = J^2 - J_z^2 + \hbar J_z$$

Proof - Eigenvalues of J^2 and positive or zero

$$\langle jm | J^2 | jm \rangle = j(j+1)\hbar^2$$

$$= \langle jm | J_x^2 | jm \rangle + \langle jm | J_y^2 | jm \rangle + \langle jm | J_z^2 | jm \rangle$$

$$= | J_x | jm \rangle |^2 + | J_y | jm \rangle |^2 + | J_z | jm \rangle |^2$$

$$\geq 0$$

using J_i Hermitian and $| |a\rangle | \geq 0$.

Proof - m satisfies the relation $-j \leq m \leq j$

$$| J_+ | jm \rangle |^2 = \langle jm | J_- J_+ | jm \rangle \geq 0$$

$$= \langle jm | J^2 - J_z^2 - \hbar J_z | jm \rangle \geq 0$$

$$= j(j+1)\hbar^2 - m^2\hbar^2 - m\hbar^2 \geq 0$$

$$j(j+1) \geq m(m+1)$$

$$j \geq m$$

Likewise

$$|J_-|jm\rangle|^2 = \langle jm|J_+J_-|jm\rangle \geq 0$$

$$= \langle jm|J^2 - J_z^2 + \hbar J_z|jm\rangle \geq 0$$

$$j(j+1)\hbar^2 - m^2\hbar^2 + m\hbar^2 \geq 0$$

$$j(j+1) - m(m-1) \geq 0$$

Suppose $m < 0$, $\quad m = -|m|$

$$j(j+1) - |m|(1+|m|) \geq 0$$

$$j(j+1) \geq +|m|(|m|+1)$$

Suppose $m < -j$

$$j(j+1) \geq |m|(|m|+1) > |j|(|j|+1)$$

so $-j \leq m$

Lower $|j\,-j\rangle$ gives zero

$$\left|\hat{J}_-\,|j,-j\rangle\right|^2 = \langle j\,-j|\,\hat{J}_+\hat{J}_-\,|j\,-j\rangle$$

$$= j(j+1)\hbar^2 - j^2\hbar^2 - j\hbar^2$$

$$= 0 \quad \checkmark$$

Raising $|jj\rangle$ gives zero

$$\left|J_+\,|jj\rangle\right|^2 = \langle jj\,|J_-J_+|\,jj\rangle$$

$$= \hbar^2\left(j(j+1) - j^2 - j\right) = 0 \quad \checkmark$$

$J_-|jm\rangle = c\,|j\,m-1\rangle$ if $m > -j$

where c is a constant

$$[J^2, J_-]\,|jm\rangle = 0$$

$$J^2J_-\,|jm\rangle - J_-\mathbf{J}^2|jm\rangle = 0$$

$$J^2\left(J_-|jm\rangle\right) = j(j+1)\hbar^2\left(J_-|jm\rangle\right)$$

$\Rightarrow$ $J_-|jm\rangle$ is an eigenvector of J^2 with eigenvalue $j(j+1)\hbar^2$.

$\Rightarrow$ J_- does not change j.

Now try,

$$[J_z, J_-]|jm\rangle = J_z J_-|jm\rangle - J_- J_z|jm\rangle$$

$$= -\hbar J_-|jm\rangle \quad \left(\begin{array}{c}\text{Property of} \\ \text{commutator}\end{array}\right)$$

$$J_z\left(J_-|jm\rangle\right) - m\hbar J_-|jm\rangle$$

$$= -\hbar J_-|jm\rangle$$

$$J_z\left(J_-|jm\rangle\right) = (m-1)\hbar\left(J_-|jm\rangle\right)$$

$\Rightarrow$ $J_-|jm\rangle$ is an eigenvector of J_z with eigenvalue $(m-1)\hbar$ $\Rightarrow$ $J_-|jm\rangle = c|jm-1\rangle$, where c is a constant.

The m levels are spaced 1 apart and trapped between $-j$ and j

$$-j \le m \le j$$

$$\Longrightarrow \quad j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2} \cdots$$

otherwise starting at $|j,j\rangle$ does not get you to $|j\,-j\rangle$.

So as suspected, J_- is a lowering operator.

<u>Likewise</u> $\quad J_+|j\,m\rangle = c\,|j\,m+1\rangle$

if $m \neq j$.

$\Rightarrow$ J_+ is the raising operator.

We need to fix c.

<u>Normalized Ladder Operator</u>

$$|j\;m+1\rangle = \frac{\hat{J}_+}{\hbar\sqrt{j(j+1)-m(m+1)}}\,|j\,m\rangle$$

$$|j\;m-1\rangle = \frac{\hat{J}_-}{\hbar\sqrt{j(j+1)-m(m-1)}}\,|j\,m\rangle$$