Clebsch - Gordan

In general the total angular momentum of two particles with momentum s_1 and s_2 runs from $s_1 + s_2$ to $|s_1 - s_2|$ in integer steps.

In our previous work we found that when two spin $1/2$ particles are added, the resulting angular momentum extended from $1 = \frac{1}{2} + \frac{1}{2}$ to $0 = \left|\frac{1}{2} - \frac{1}{2}\right|$.

We also found the expansion of the combinated states $|s\, m\rangle$ in terms of the single particle states $|s_1 m_1\rangle |s_2 m_2\rangle$

$$|1\,1\rangle = |\tfrac{1}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle$$

$$|1\,0\rangle = \tfrac{1}{\sqrt{2}}|\tfrac{1}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{2}}|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle$$

$$|1\,-1\rangle = |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$$

$$|0\,0\rangle = \tfrac{1}{\sqrt{2}}|\tfrac{1}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle - \tfrac{1}{\sqrt{2}}|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle$$

We might also need the single particle states in terms of the combined states.

$$|\tfrac{1}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle = |1\,1\rangle$$

$$|\tfrac{1}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{2}}|1\,0\rangle + \tfrac{1}{\sqrt{2}}|0\,0\rangle$$

$$|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{2}}|1\,0\rangle - \tfrac{1}{\sqrt{2}}|0\,0\rangle$$

$$|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle = |1\,-1\rangle$$

We figured those out by guessing $|ss\rangle$, lowering, and then guessing an orthogonal vector and checking it. This will become remarkably tedious as s_1, s_2 become larger.

What we have done in general is to work out the coefficients to two expansions:

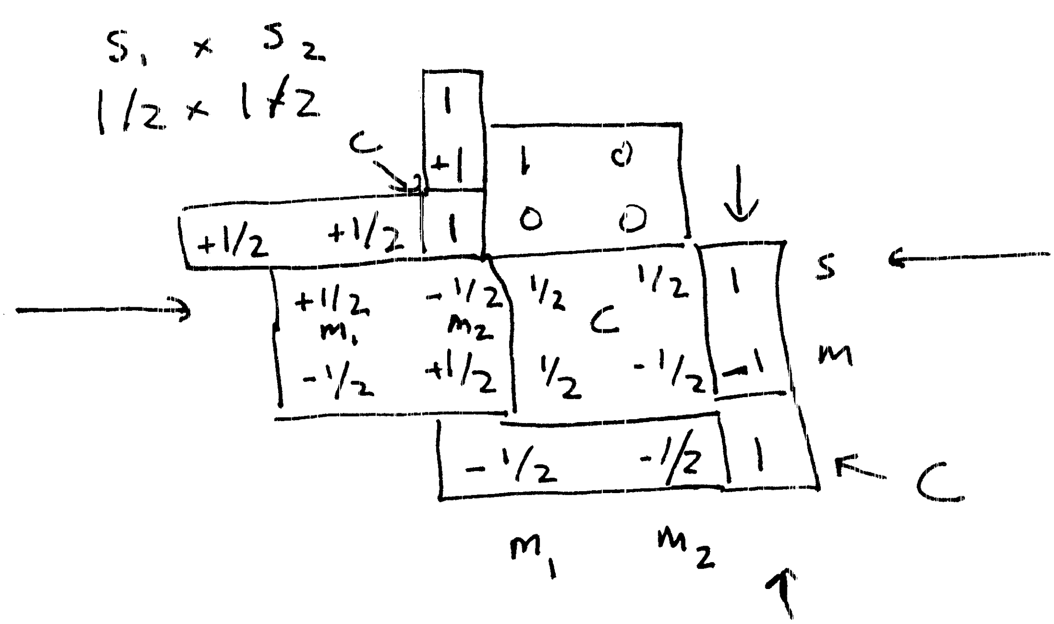
Total spin in terms of individual spins:

$$|s\,m\rangle = \sum_{m_1, m_2} C^{s_1\, s_2\, s}_{m_1\, m_2\, m} |s_1\, m_1\rangle |s_2\, m_2\rangle$$

Individual spin in terms of total spin

$$|s_1\, m_1\rangle |s_2\, m_2\rangle = \sum_{s, m} C^{s_1\, s_2\, s}_{m_1\, m_2\, m} |s\, m\rangle$$

The coefficients $C^{s_1 s_2 s}_{m_1 m_2 m}$ are called Clebsch-Gordon coefficients and are generally read from tables. Consider the $1/2 \times 1/2$ table



Consider the row indicated

$$\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle \left|\tfrac{1}{2}\ {-\tfrac{1}{2}}\right\rangle = \frac{1}{\sqrt{2}}\left|1\,0\right\rangle + \frac{1}{\sqrt{2}}\left|0\,0\right\rangle$$

Note a $\sqrt{}$ is implied, so $-1/2 \to -\frac{1}{\sqrt{2}}$.

Consider the column indicated

$$|1\,-1\rangle = 1 \cdot |\tfrac{1}{2}\,-\tfrac{1}{2}\rangle|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$$

Ex A spin $s_1 = 2$ particle is combined with a spin $s_2 = 1$ particle. The system is observed to be in the total angular momentum state $s = 3$, $m = 2$. What is the probability of measuring the $s = 2$ spin particle in its highest S_z state.

S/n Look at the 2×1 table. Total states are columns, find the $3, 2$ column.

$$|3\,2\rangle = \sqrt{\tfrac{1}{3}}\,|2\,2\rangle|1\,0\rangle + \sqrt{\tfrac{2}{3}}\,|2\,1\rangle|1\,1\rangle$$

$$P(m_1 = 2) = \left(\sqrt{\tfrac{1}{3}}\right)^2 = \frac{1}{3}$$

34. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2			
m_1	m_2	Coefficients		
.	.			
.	.			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

1/2×1/2

M	m_1	m_2	J=1	J=0
+1	+1/2	+1/2	1	
0	+1/2	−1/2	1/2	1/2
0	−1/2	+1/2	1/2	−1/2
−1	−1/2	−1/2	1	

1×1/2

M	m_1	m_2	J=3/2	J=1/2
+3/2	+1	+1/2	1	
+1/2	+1	−1/2	1/3	2/3
+1/2	0	+1/2	2/3	−1/3
−1/2	0	−1/2	2/3	1/3
−1/2	−1	+1/2	1/3	−2/3
−3/2	−1	−1/2	1	

2×1/2

M	m_1	m_2	J=5/2	J=3/2
+5/2	+2	+1/2	1	
+3/2	+2	−1/2	1/5	4/5
+3/2	+1	+1/2	4/5	−1/5
+1/2	+1	−1/2	2/5	3/5
+1/2	0	+1/2	3/5	−2/5
−1/2	0	−1/2	3/5	2/5
−1/2	−1	+1/2	2/5	−3/5
−3/2	−1	−1/2	4/5	1/5
−3/2	−2	+1/2	1/5	−4/5
−5/2	−2	−1/2	1	

3/2×1/2

M	m_1	m_2	J=2	J=1
+2	+3/2	+1/2	1	
+1	+3/2	−1/2	1/4	3/4
+1	+1/2	+1/2	3/4	−1/4
0	+1/2	−1/2	1/2	1/2
0	−1/2	+1/2	1/2	−1/2
−1	−1/2	−1/2	3/4	1/4
−1	−3/2	+1/2	1/4	−3/4
−2	−3/2	−1/2	1	

2×1

M	m_1	m_2	J=3	J=2	J=1
+3	+2	+1	1		
+2	+2	0	1/3	2/3	
+2	+1	+1	2/3	−1/3	
+1	+2	−1	1/15	1/3	3/5
+1	+1	0	8/15	1/6	−3/10
+1	0	+1	2/5	−1/2	1/10
0	+1	−1	1/5	1/2	3/10
0	0	0	3/5	0	−2/5
0	−1	+1	1/5	−1/2	3/10
−1	0	−1	2/5	1/2	1/10
−1	−1	0	8/15	−1/6	−3/10
−1	−2	+1	1/15	−1/3	3/5
−2	−1	−1	2/3	1/3	
−2	−2	0	1/3	−2/3	
−3	−2	−1	1		

3/2×1

M	m_1	m_2	J=5/2	J=3/2	J=1/2
+5/2	+3/2	+1	1		
+3/2	+3/2	0	2/5	3/5	
+3/2	+1/2	+1	3/5	−2/5	
+1/2	+3/2	−1	1/10	2/5	1/2
+1/2	+1/2	0	3/5	1/15	−1/3
+1/2	−1/2	+1	3/10	−8/15	1/6
−1/2	+1/2	−1	3/10	8/15	1/6
−1/2	−1/2	0	3/5	−1/15	−1/3
−1/2	−3/2	+1	1/10	−2/5	1/2
−3/2	−1/2	−1	3/5	2/5	
−3/2	−3/2	0	2/5	−3/5	
−5/2	−3/2	−1	1		

1×1

M	m_1	m_2	J=2	J=1	J=0
+2	+1	+1	1		
+1	+1	0	1/2	1/2	
+1	0	+1	1/2	−1/2	
0	+1	−1	1/6	1/2	1/3
0	0	0	2/3	0	−1/3
0	−1	+1	1/6	−1/2	1/3
−1	0	−1	1/2	1/2	
−1	−1	0	1/2	−1/2	
−2	−1	−1	1		

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

3/2×3/2

M	m_1	m_2	J=3	J=2	J=1	J=0
+3	+3/2	+3/2	1			
+2	+3/2	+1/2	1/2	1/2		
+2	+1/2	+3/2	1/2	−1/2		
+1	+3/2	−1/2	1/5	1/2	3/10	
+1	+1/2	+1/2	3/5	0	−2/5	
+1	−1/2	+3/2	1/5	−1/2	3/10	
0	+3/2	−3/2	1/20	1/4	9/20	1/4
0	+1/2	−1/2	9/20	1/4	−1/20	−1/4
0	−1/2	+1/2	9/20	−1/4	−1/20	1/4
0	−3/2	+3/2	1/20	−1/4	9/20	−1/4
−1	+1/2	−3/2	1/5	1/2	3/10	
−1	−1/2	−1/2	3/5	0	−2/5	
−1	−3/2	+1/2	1/5	−1/2	3/10	
−2	−1/2	−3/2	1/2	1/2		
−2	−3/2	−1/2	1/2	−1/2		
−3	−3/2	−3/2	1			

$$d^1_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^1_{1,1} = \frac{1+\cos\theta}{2}$$

$$d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^1_{1,-1} = \frac{1-\cos\theta}{2}$$

2×3/2

M	m_1	m_2	J=7/2	J=5/2	J=3/2	J=1/2
+7/2	+2	+3/2	1			
+5/2	+2	+1/2	3/7	4/7		
+5/2	+1	+3/2	4/7	−3/7		
+3/2	+2	−1/2	1/7	16/35	2/5	
+3/2	+1	+1/2	4/7	1/35	−2/5	
+3/2	0	+3/2	2/7	−18/35	1/5	
+1/2	+2	−3/2	1/35	6/35	2/5	2/5
+1/2	+1	−1/2	12/35	5/14	0	−3/10
+1/2	0	+1/2	18/35	−3/35	−1/5	1/5
+1/2	−1	+3/2	4/35	−27/70	2/5	−1/10
−1/2	+1	−3/2	4/35	27/70	2/5	1/10
−1/2	0	−1/2	18/35	3/35	−1/5	−1/5
−1/2	−1	+1/2	12/35	−5/14	0	3/10
−1/2	−2	+3/2	1/35	−6/35	2/5	−2/5
−3/2	0	−3/2	2/7	18/35	1/5	
−3/2	−1	−1/2	4/7	−1/35	−2/5	
−3/2	−2	+1/2	1/7	−16/35	2/5	
−5/2	−1	−3/2	4/7	3/7		
−5/2	−2	−1/2	3/7	−4/7		
−7/2	−2	−3/2	1			

2×2

M	m_1	m_2	J=4	J=3	J=2	J=1	J=0
+4	+2	+2	1				
+3	+2	+1	1/2	1/2			
+3	+1	+2	1/2	−1/2			
+2	+2	0	3/14	1/2	2/7		
+2	+1	+1	4/7	0	−3/7		
+2	0	+2	3/14	−1/2	2/7		
+1	+2	−1	1/14	3/10	3/7	1/5	
+1	+1	0	3/7	1/5	−1/14	−3/10	
+1	0	+1	3/7	−1/5	−1/14	3/10	
+1	−1	+2	1/14	−3/10	3/7	−1/5	
0	+2	−2	1/70	1/10	2/7	2/5	1/5
0	+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	0	18/35	0	−2/7	0	1/5
0	−1	+1	8/35	−2/5	1/14	1/10	−1/5
0	−2	+2	1/70	−1/10	2/7	−2/5	1/5
−1	+1	−2	1/14	3/10	3/7	1/5	
−1	0	−1	3/7	1/5	−1/14	−3/10	
−1	−1	0	3/7	−1/5	−1/14	3/10	
−1	−2	+1	1/14	−3/10	3/7	−1/5	
−2	0	−2	3/14	1/2	2/7		
−2	−1	−1	4/7	0	−3/7		
−2	−2	0	3/14	−1/2	2/7		
−3	−1	−2	1/2	1/2			
−3	−2	−1	1/2	−1/2			
−4	−2	−2	1				

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^2_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^2_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2$$

$$d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^2_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Figure 34.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.