Complex Numbers

A complex number is a pair of real numbers.

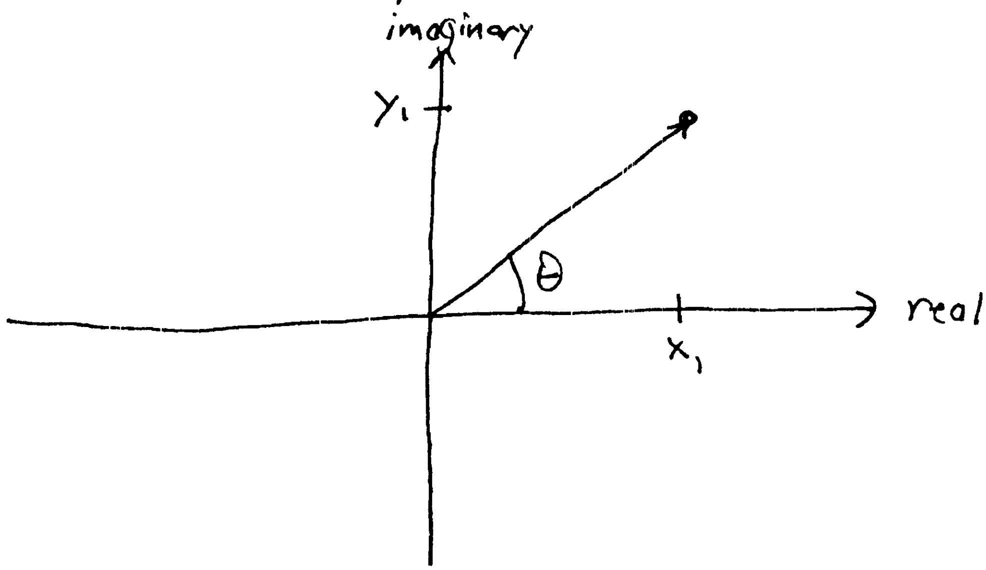
$$(x, y)$$

where the first component, x, is called the real part and the second component, y, is called the imaginary part.

For additive operations, the complex number behaves just like a two dimensional vector. If $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ then

$$z = z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$$

We can plot the complex number as a vector in a two dimensional plane



We can take the length of the vector, $|z|$

$$|z| = \sqrt{x^2+y^2} \equiv \rho$$

and decompose the vector into components

$$z = \rho(\cos\theta, \sin\theta) \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Complex numbers behave very differently than vectors under multiplication.

$$\boxed{\text{Define} \quad i = \sqrt{-1} \qquad i^2 = -1}$$

Write $\quad z = x + iy = (x, y)$

$$z^2 = (x+iy)^2 = x^2 + 2ixy + (iy)^2$$

$$= x^2 - y^2 + 2ixy$$

$$= (x^2-y^2, 2xy)$$

Complex conjugate z^* - replace all i's by $-i$

If $z = x + iy$, $z^* = x - iy$.

Consider, $z^* z = (x-iy)(x+iy)$ $\Longrightarrow$ Real

$$= x^2 - ixy + ixy - i^2 y^2$$
$$= x^2 + y^2 = \rho^2$$

<u>Modulus</u> $\rho = \sqrt{z^* z}$

<u>Euler's Formula</u>

$$e^{i\theta} = \cos\theta + i\sin\theta$$

* Prove by examining the power series.
* One of the most useful formulas in mathematics

If $z = \rho\cos\theta + i\rho\sin\theta = \rho e^{i\theta}$

for any complex number

$$z_1 z_2 = \rho_1 \rho_2 e^{i\theta_1} e^{i\theta_2}$$
$$= \rho_1 \rho_2 e^{i(\theta_1+\theta_2)}$$

If a relation can be proven for a complex number, like $f(z_1) = z_0$, then the real and imaginary parts of the relation must be equal.

If $z = a + ib$

Real Part $\quad \mathrm{Re}(z) = a$

Imaginary Part $\quad \mathrm{Im}(z) = b$

$$\mathrm{Re}(f(z_1)) = \mathrm{Re}(z_0)$$

$$\mathrm{Im}(f(z_1)) = \mathrm{Im}(z_0)$$

Example Prove double angle formulas

$$e^{2i\theta} = e^{i\theta} e^{i\theta} = (\cos\theta + i\sin\theta)^2$$

$$= \cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta$$

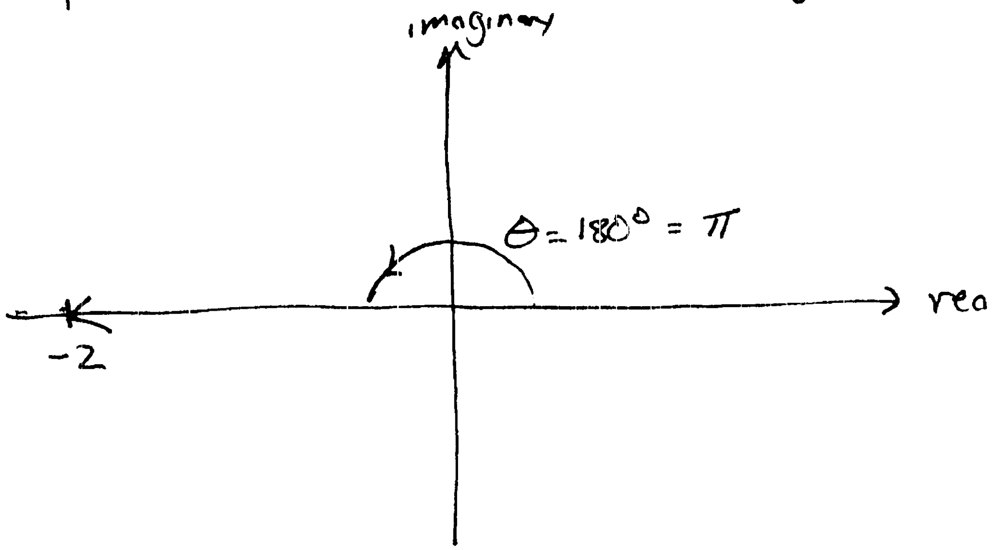
but $e^{2i\theta} = \cos 2\theta + i\sin 2\theta$

The real and imaginary parts must be equal.

$$\cos 2\theta = \cos^2\theta - \sin^2\theta \qquad \text{real}$$

$$\sin 2\theta = 2\cos\theta\sin\theta \qquad \text{imaginary}$$

Example Represent -2 as $\rho e^{i\theta}$



$$\rho = \sqrt{z^* z} = \sqrt{(-2)(-2)} = 2$$

$$-1 = e^{i\pi} = \underbrace{\cos\pi}_{-1} + \underbrace{i\sin\pi}_{0}$$

$$-2 = 2e^{i\pi}$$

Example Represent $\dfrac{i}{1+i} = z$

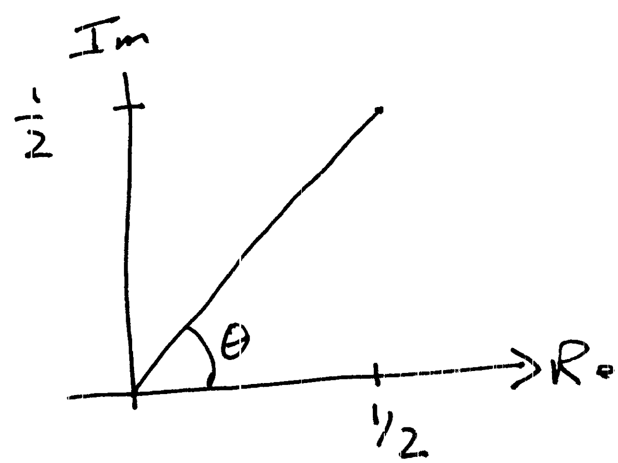
as $a+ib$ and $\rho e^{i\theta}$

Sln Get i out of the denominator by multiplying by ~~z~~ complex conjugate

$$z = \frac{i}{1+i} = \frac{i}{1+i} \cdot \frac{1-i}{1-i}$$

$$= \frac{i+1}{2} = \frac{1}{2} + i\frac{1}{2}$$

$$\rho = \sqrt{z z^*} = \sqrt{\frac{i}{1+i} \cdot \frac{-i}{1-i}} = \sqrt{\frac{1}{2}}$$



$$\tan\theta = \frac{1/2}{1/2} = \frac{b}{a} = 1$$

$$\theta = \frac{\pi}{4}$$

$$z = \sqrt{\frac{1}{2}}\, e^{i\pi/4}$$

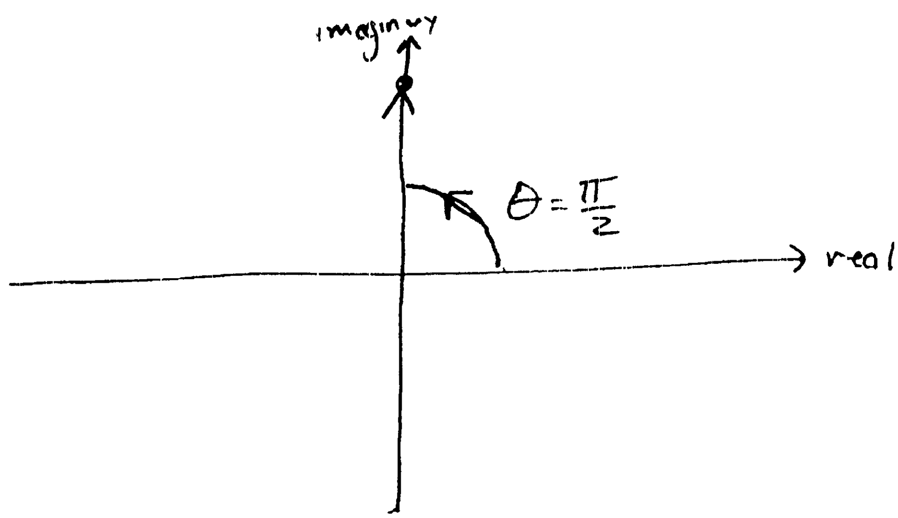
Example - Represent $\frac{1+i}{1-i}$ as $\rho e^{i\theta}$

$$\rho = \sqrt{z^* z} = \sqrt{\frac{(1+i)(1-i)}{(1-i)(1+i)}} = 1$$

Plot as vector to get θ - Problem - must get i out of denominator. Multiply by complex conjugate to get real number

$$\frac{1+i}{1-i} \cdot \frac{1+i}{1+i} = \frac{1+2i-1}{1+1}$$

$$= \frac{2i}{1+1} = i$$



$$\frac{1+i}{1-i} = i = e^{i\pi/2}$$

Example Find $\sqrt{i}$

What does $\sqrt{\ }$ do to complex numbers?

What does it do to real exponentials?

$$\sqrt{e^x} = e^{x/2}$$

$$\sqrt{z} = \sqrt{\rho e^{i\theta}} = \sqrt{\rho}\, e^{i\theta/2}$$

So $\quad i = e^{i\pi/2}$

$$\sqrt{i} = e^{i\pi/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

More Euler

Since $e^{i\theta} = \cos\theta + i\sin\theta$

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$$

$$= \cos(\theta) - i\sin\theta$$

So $e^{i\theta} + e^{-i\theta} = 2\cos\theta$

and $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$