Complete Sets of Commuting Observables

Hermitian Operators

- Eigenvecters with different eigenvalues orthogonal.
- Eigenvalues real.
- Observable if eigenvectors span the space.

If two operators commute, $[\hat{A}, \hat{B}] = 0$, then the an eigenvector of one is an eigenvector of the other.

(a) If the eigenvalue is <u>non-degenerate</u>, then the eigenvector is the same.

(b) If the eigenvalue of $\hat{A}$ is <u>degenerate</u> and associated with a set of eigenvectors $\{|a_i\rangle\}$, then the action of $\hat{B}$ does not take a vector in the subspace spanned by $\{|a_i\rangle\}$ out of the subspace.

(c) If $[\hat{A}, \hat{B}] = 0$, there exists a basis where both are diagonal.

In a basis of its eigenvectors, a matrix is diagonal.

Example The eigenvalues of the $\hat{L}_z$ matrix for $l=1$ are $0, \pm\hbar$. In the basis of the eigenvectors of L_z,

$$\hat{L}_z = \begin{pmatrix} \hbar & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\hbar \end{pmatrix}$$

Vocabulary

Spectrum of Operator - Set of eigenvalues.

Non-degenerate Eigenvalue - A value with only one eigenvector.

Degenerate Eigenvalue - An eigenvalue associated with more than one eigenvector.

Diagonal Matrix - A matrix with all non-zero elements on the diagonal.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

<u>Block Diagonal Matrix</u> A matrix where all non-zero elements are in square blocks along diagonal.

$$\begin{pmatrix} a & b & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 \\ 0 & 0 & e & 0 & 0 \\ 0 & 0 & 0 & f & g \\ 0 & 0 & 0 & h & i \end{pmatrix}$$

~~If $\hat{A}$ has block diag~~

If we are working in the basis of the eigenvectors of $\hat{A}$

$$\hat{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

and if $[\hat{A}, \hat{B}] = 0$ then

$$\hat{B} = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 & 4 & 0 \\ 0 & 0 & 3 & 4 & 5 & 0 \\ 0 & 0 & 4 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix}$$

$\hat{B}$ only mixes the subspace of the degenerate eigenvectors.

We can form a new basis with ~~the~~ the Gram-Schmidt process in the degenerate subspace that diagonalizes $\hat{A}$ and $\hat{B}$ at the same time. If the eigenvalues of $\hat{B}$ are non-degenerate in each degenerate subspace we can uniquely label the basis vectors of the space as $|ab\rangle$.

Complete Set of Commuting Observables (CSCO)

A set of observables that commute in pairs such that the set of eigenvalues uniquely determine the basis vector.

$\Rightarrow$ Measurement of one member of a CSCO does not modify the measurement of another.

$\Rightarrow$ Sequential measurement of a CSCO uniquely determine the state of the system.