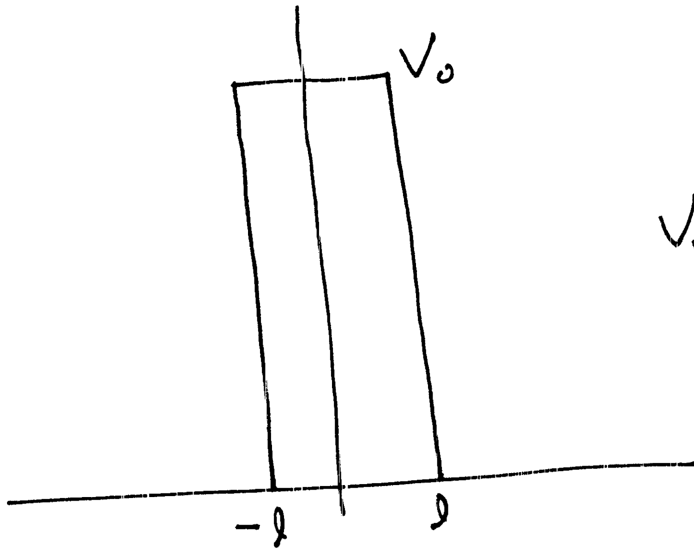
Delta Function Potentials

Consider a narrow barrier



$$V_r = \begin{cases} 0 & \text{otherwise} \\ V_0 & x \in [-l, l] \end{cases}$$

Real function

If we don't know the details of the potential within the barrier, we could model the potential with a delta function

$$V_m(x) = \alpha \delta(x)$$ ← Model function

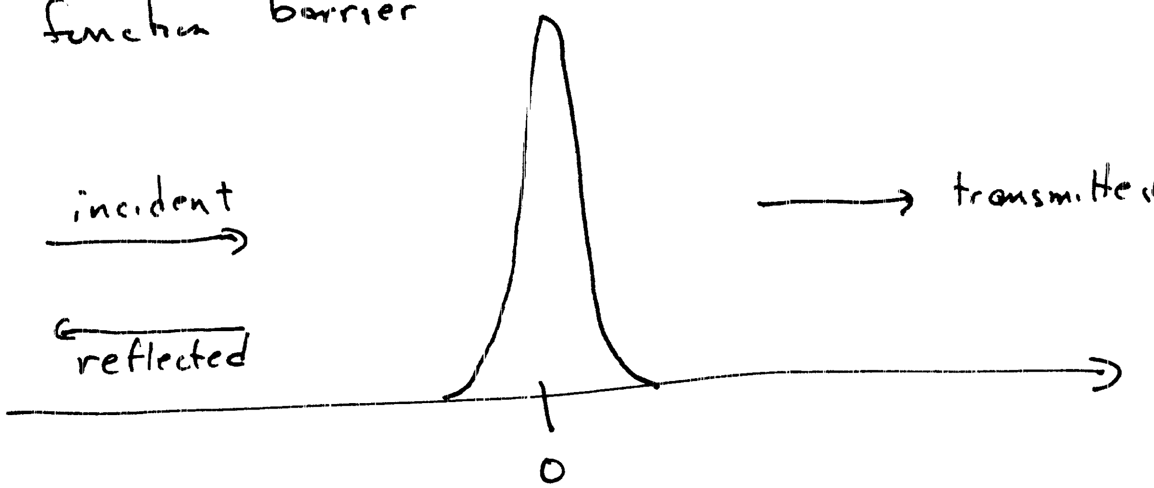
To find α, integrate both functions

$$\int_{-\infty}^{\infty} V_r \, dx = 2 l V_0 = \int_{-\infty}^{\infty} V_m(x) \, dx = \alpha$$

$$\alpha = 2 l V_0$$

α = thickness · strength of the potential

Consider a particle incident on a delta function barrier



Except at $x=0$, $V=0$ so the TISE is

$$-\frac{\hbar^2}{2m}\frac{d^2\phi}{dx^2} = E\phi$$

$$\frac{d^2\phi}{dx^2} + k^2\phi = 0 \qquad k^2 = \frac{2mE}{\hbar^2}$$

$\Rightarrow$ k of free particle

As before write solution as incident, reflected and transmitted wave

$\underline{x<0}$ $$\phi_-(x) = \phi_I + \phi_R$$
$$= A_I e^{ikx} + A_R e^{-ikx}$$
incident reflected

$\underline{x>0}$ $$\phi_+(x) = \phi_T = A_T e^{ikx}$$

Boundary Conditions

Wave function is always continuous.

$$\phi_-(0) = \phi_+(0)$$

$$A_I + A_R = A_T$$

To develop the boundary condition for the slope we integrate the TISE.

TISE

$$-\frac{\hbar^2}{2m}\frac{d^2\phi}{dx^2} + V\phi = E\phi \qquad V = \alpha\delta(x)$$

Integrate from $-\varepsilon$ to $+\varepsilon$

$$-\frac{\hbar^2}{2m}\int_{-\varepsilon}^{\varepsilon}\frac{d^2\phi}{dx^2}\,dx + \int_{-\varepsilon}^{\varepsilon}\alpha\delta(x)\,dx\,\phi(x)$$

$$= \int_{-\varepsilon}^{\varepsilon} E\phi(x)\,dx$$

$$-\frac{\hbar^2}{2m}\left[\left.\frac{d\phi}{dx}\right|_{+\varepsilon} - \left.\frac{d\phi}{dx}\right|_{-\varepsilon}\right] + \alpha\phi(0) = 2E\varepsilon\,\phi(0)$$

If $\varepsilon \to 0$, $2E\varepsilon\phi(0) \to 0$

Delta Function Boundary Condition

$$\left.\frac{d\phi}{dx}\right|_{+\varepsilon} - \left.\frac{d\phi}{dx}\right|_{-\varepsilon} = \frac{2m\alpha}{\hbar^2}\phi(0)$$

The slope is discontinuous by a factor of

$$\frac{2m\alpha}{\hbar^2}\phi(0)$$

Apply Delta Function Boundary Condition

$$\left.\frac{d\phi}{dx}\right|_- = ikA_I - ikA_R$$

$$\left.\frac{d\phi}{dx}\right|_+ = ikA_T$$

$$\left.\frac{d\phi}{dx}\right|_+ - \left.\frac{d\phi}{dx}\right|_- = ikA_T - (ikA_I - ikA_R)$$

$$= \frac{2m\alpha}{\hbar^2}\phi(0) = \frac{2m\alpha}{\hbar^2}A_T$$

<u>Divide by ik</u>

$$A_T - A_I + A_R = \frac{2m\alpha}{ik\hbar^2}A_T = -2B\,i\,A_T$$

$$B \equiv \frac{m\alpha}{\hbar^2 k}$$

Equations

$$A_I + A_R = A_T$$

$$A_T - A_I + A_R = -2iBA_T$$

Eliminate A_R

$$A_R = A_T - A_I$$

$$A_T - A_I + (A_T - A_I) = -2iBA_T$$

$$2A_T - 2A_I = -2iBA_T$$

$$A_T(1+iB) = A_I$$

$$\frac{A_T}{A_I} = \frac{1}{1+iB}$$

Solve for A_R

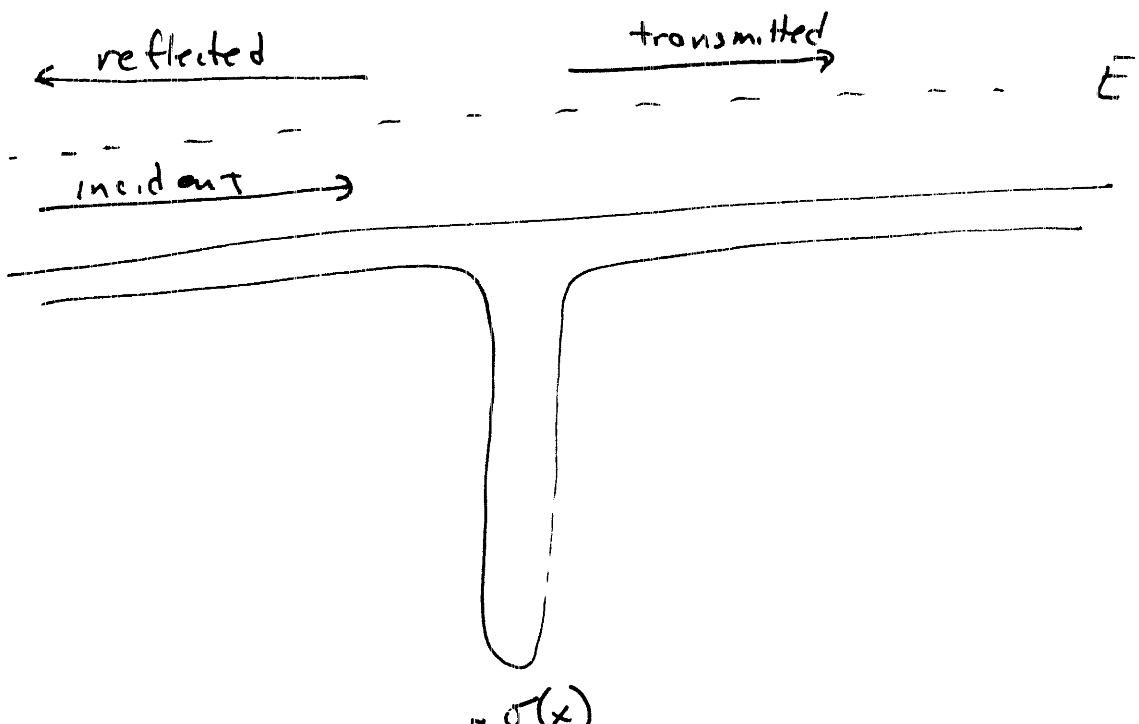
$$\frac{A_R}{A_I} = \frac{A_T}{A_I} - 1 = \frac{1}{1+iB} - 1 = \frac{-iB}{1+iB}$$

Reflection and Transmission

$$R = \frac{A_R^* A_R}{A_I^* A_I} = \frac{\beta^2}{1+\beta^2}$$

$$T = \cancel{A} \; 1 - R = \frac{1}{1+\beta^2}$$

This solution also works for a delta function well if $E > 0$, $V = -\alpha\,\delta(x)$



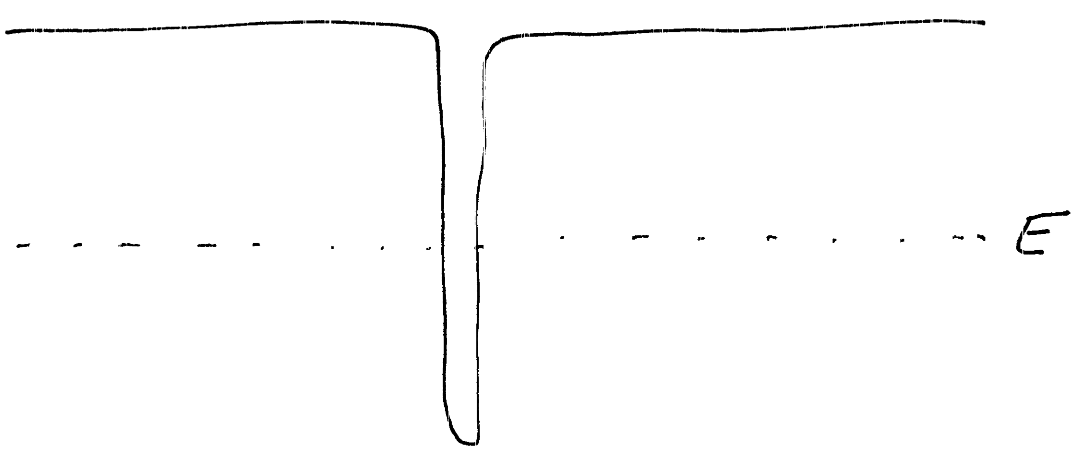
If $\alpha \rightarrow -\alpha$, $B \rightarrow -B$

$$B = \left| \frac{m\alpha}{\hbar^2 k} \right|$$

$$\frac{A_R}{A_I} = \frac{iB}{1-iB} \qquad \frac{A_T}{A_I} = \frac{1}{1-iB}$$

R, T are unchanged.

Delta Function Well $E<0$, $V=-\alpha\delta(x)$



TISE

$$-\frac{\hbar^2}{2m}\frac{d^2\phi}{dx^2} + V(x)\phi = E\phi$$

$$V(x) = -\alpha\delta(x) \qquad \alpha > 0$$

If $x \neq 0$,

$$-\frac{\hbar^2}{2m}\frac{d^2\phi}{dx^2} = E\phi \qquad E<0$$

$$\frac{d^2\phi}{dx^2} - k^2\phi = 0$$

$$k = \sqrt{\frac{-2mE}{\hbar^2}}$$

Solutions $\quad \phi = e^{\pm kx}$

Because the wave function must be normalizable,

$$\phi(x) = \begin{cases} A_- e^{\kappa x} & x < 0 \\ A_+ e^{-\kappa x} & x > 0 \end{cases}$$

Boundary Conditions

Continuous

$$A_- = A_+ \equiv A$$

Slope

$$\left.\frac{d\phi}{dx}\right|_+ - \left.\frac{d\phi}{dx}\right|_- = \frac{-2m\alpha}{\hbar^2}\phi(0) = \frac{-2m\alpha}{\hbar^2} A$$

$$-\kappa A \quad - \kappa A \quad =$$

$$2A\kappa = \frac{2m\alpha}{\hbar^2} A$$

$$\kappa = \frac{m\alpha}{\hbar^2}$$

Energy

$$E = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 (ik)^2}{2m}$$

$$= -\frac{\hbar^2 \kappa^2}{2m} = -\frac{\hbar^2}{2m}\left(\frac{m\alpha}{\hbar^2}\right)^2$$

$$= -\frac{m\alpha^2}{2\hbar^2}$$