Dirac Notation

We have been using a few pieces of Dirac notation: $|a\rangle$ as a ket and $\langle b|a\rangle$ as $I(|b\rangle, |a\rangle)$. What then is $\langle b|$?

By definition, $I(|b\rangle, |a\rangle) = \langle b|a\rangle$, so if we remove $|a\rangle$ we have

$$I(|b\rangle, -) = \langle b|$$

$\Rightarrow$ The vector $|b\rangle$ <u>AND</u> the machinery that does the inner product.

$\Rightarrow$ For example, if $\langle b|a\rangle = \vec{b}\cdot\vec{a}$ the $\langle b| = \vec{b}\,\cdot$

This lets us do pure magic.

Suppose $|x\rangle \equiv \hat{x}$, the $\langle x| = \hat{x}\cdot$

We can define a linear operator $\hat{T}$ by

$$\hat{T} = |x\rangle\langle x| = \hat{x}(\hat{x}\cdot)$$

Let's let $\hat{T}$ act on a vector

$$\hat{T}\vec{E} = \hat{T}(E_1\hat{x} + E_2\hat{y} + E_3\hat{z})$$

$$= \hat{x}(\hat{x}\cdot)(E_1\hat{x} + E_2\hat{y} + E_3\hat{z})$$

$$= \hat{x}(E_1\hat{x}\cdot\hat{x} + E_2\hat{x}\cdot\hat{y} + E_3\hat{x}\cdot\hat{z})$$

$$= E_1\hat{x}$$

The operator $\hat{T}$ projected $\vec{E}$ onto $\hat{x}$.

We will called $\langle b|$ a bra, so

$\langle b|a\rangle$
bra ket

Now consider $\langle b|\hat{A}|c\rangle$ which could mean $\langle b|(\hat{A}|c\rangle)$ or $(\langle b|\hat{A})|a\rangle$.

The first is straight ~~forward~~ since $\hat{A}|c\rangle$ is simply another vector. We want to define the second one so both expressions ~~are~~ equal.

Resert to the definition,

$$I(|b\rangle, \hat{A}|c\rangle) = \langle b|(\hat{A}|c\rangle)$$

$$= I(\hat{A}^{+}|b\rangle, |c\rangle) \text{ by definition of adjoint.}$$

Therefore, let $(\langle b|\hat{A})$ be bra ~~formed~~ from the vector $\hat{A}^{+}|b\rangle$. ~~There~~ With this definition, $\langle b|\hat{A}|a\rangle$ means operate to the right ~~left~~ with $\hat{A}$ or to the ~~right~~ left with $\hat{A}^{+}$.

Define $|a\rangle^\dagger = \langle a|$ and $c^\dagger = c^*$.

To take the adjoint of a bunch of stuff,

1. take the adjoint of each piece.
2. adjoint of a number is its complex conjugate.
3. Reverse the order.

Example $\left(c\langle a|\hat{B}\hat{A}|b\rangle\right)^\dagger = c^*\langle b|\hat{A}^\dagger\hat{B}^\dagger|a\rangle$

Let's use dirac notation to write some useful operators.

Closure (completeness) Relation - If $\{|u_i\rangle\}$ is a complete orthonormal basis for $\mathbb{V}$ then

$$\hat{1} = \sum_i |u_i\rangle\langle u_i|$$

$$\hat{1}|a\rangle = |a\rangle$$

<u>Projection Operator</u> - If $\{|s_i\rangle\}$ is an orthonormal basis for a subspace of $\mathbb{V}$, then the operator $\hat{\mathbb{P}} = \sum_i |s_i\rangle\langle s_i|$ projects a vector into the subspace.