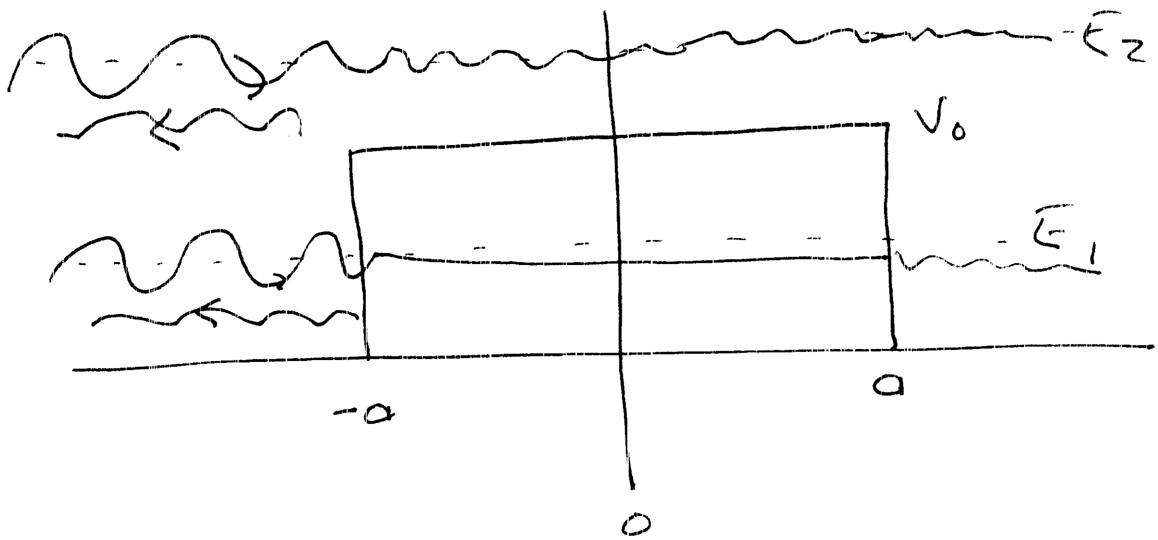
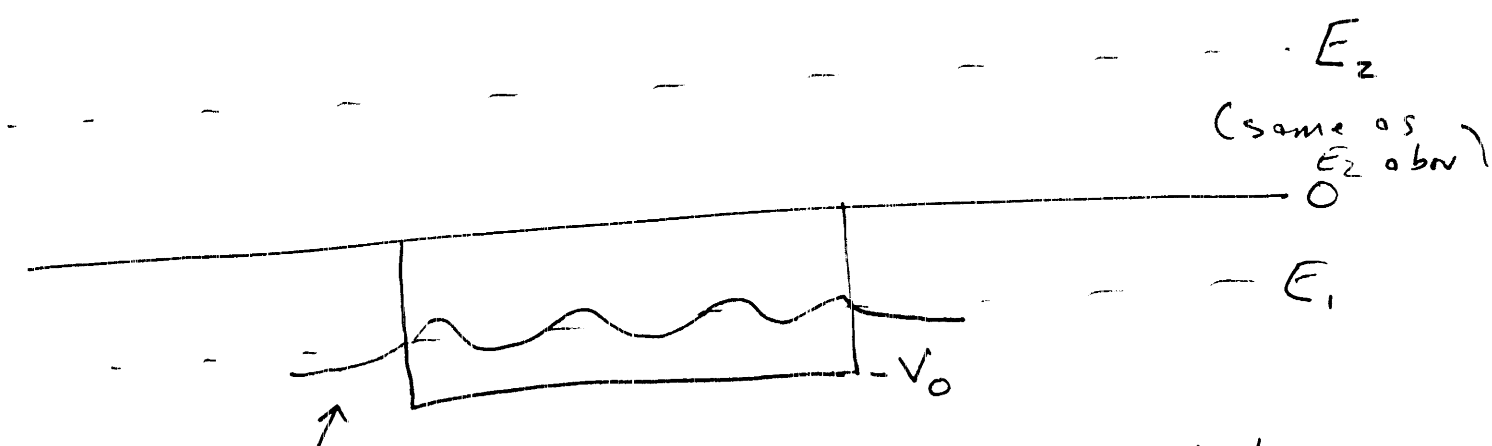
Finite Wells + Barriers

Finite Barrier



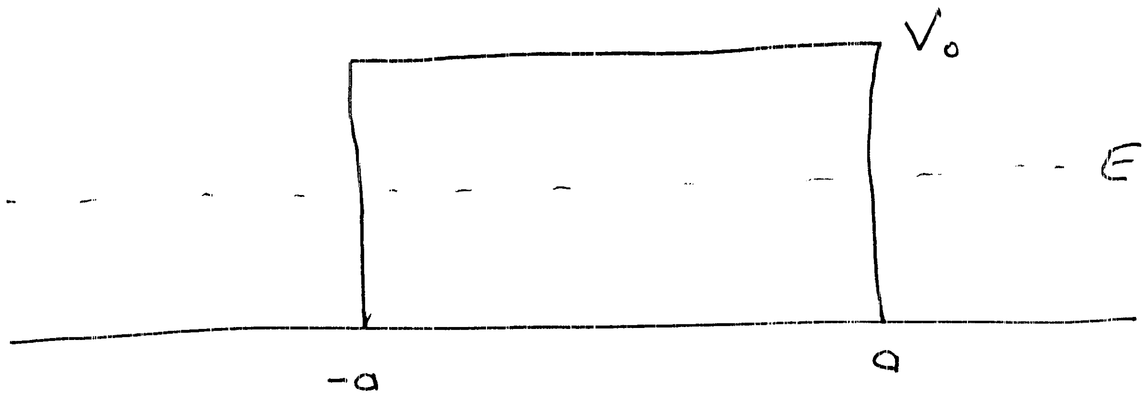
Finite Well



Some chance of finding particle outside of well.

<u>Tunneling</u> At $E < V_0$, there is some chance the particle passes through the finite barrier,

$$T \neq 0$$



<u>Solution to TISE</u>

$$\phi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x < -a \\ Ce^{\gamma x} + De^{-\gamma x} & -a < x < a \\ Fe^{ikx} & a < x \end{cases}$$

$$k = \sqrt{\frac{2mE}{\hbar^2}} \qquad \gamma = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

Boundary Conditions

Continuous at $x = -a$

(1) $$Ae^{-ika} + Be^{ika} = Ce^{-la} + De^{la}$$

Slope Continuous $x = -a$

(2) $$ikAe^{-ika} - ikBe^{ika} = lCe^{-la} - lDe^{la}$$

Continuous at $x = a$

(3) $$Ce^{la} + De^{-la} = Fe^{ika}$$

Slope Continuous at $x = a$

(4) $$lCe^{la} - lDe^{-la} = ikFe^{ika}$$

Solve

$l \cdot 3 + 4$

$$2Cle^{la} = (l+ik)Fe^{ika}$$

$$5 \qquad 2Ce^{la} = \left(1 + \frac{ik}{l}\right)Fe^{ika}$$

$-l \cdot 3 + 4$

$$-2Dle^{-la} = (ik - l)Fe^{ika}$$

$$6 \qquad 2De^{-la} = \left(1 - \frac{ik}{l}\right)Fe^{ika}$$

$ik \cdot ① + ②$

$$2ikAe^{-ika} = C(ik+l)e^{-la} + D(ik-l)e^{la}$$

$$2Ae^{-ika} = C\left(1+\frac{l}{ik}\right)e^{-la} + D\left(1-\frac{l}{ik}\right)e^{la}$$

$$⑦ \quad 2Ae^{-ika} = C\left(1-\frac{il}{k}\right)e^{-la} + D\left(1+\frac{il}{k}\right)e^{la}$$

Substitute ⑤, ⑥ into ⑦

$$2Ae^{-ika} = \frac{F}{2}\left[\left(1-\frac{il}{k}\right)\left(1+\frac{ik}{l}\right)e^{-la}e^{-la}e^{ika} + \left(1+\frac{il}{k}\right)\left(1-\frac{ik}{l}\right)e^{la}e^{la}e^{ika}\right]$$

Solve

$$A = \frac{F}{4}e^{2ika}\left[\left(1+1-\frac{il}{k}+\frac{ik}{l}\right)e^{-2la} + \left(1+1+\frac{il}{k}-\frac{ik}{l}\right)e^{2la}\right]$$

$$A = \frac{F e^{2ika}}{4}\left[2\left(e^{-2\ell a} + e^{2\ell a}\right) + i\left(\frac{\ell}{k} - \frac{k}{\ell}\right)\left(e^{2\ell a} - e^{-2\ell a}\right)\right]$$

$$= \frac{F e^{2ika}}{4}\left[4\cosh(2\ell a) + 2i\sinh(2\ell a)\left(\frac{\ell^2 - k^2}{k\ell}\right)\right]$$

where $\sinh x = \dfrac{e^{x} - e^{-x}}{2}$ $\quad \cosh x = \dfrac{e^{x} + e^{-x}}{2}$

Transmission Coefficient (Returns to same speed)

$$\frac{1}{T} = \frac{F^*F}{A^*A} = \cosh^2(2\ell a) + \left(\frac{\ell^2 - k^2}{2k\ell}\right)^2 \sinh^2(2\ell a)$$

Use $\cosh^2 x = 1 + \sinh^2 x$

$$\frac{1}{T} = 1 + \left[1 + \frac{\left(\ell^2 - k^2\right)^2}{(2k\ell)^2}\right]\sinh^2(2\ell a)$$

$$\frac{1}{T} = 1 + \left[\frac{4k^2\ell^2 + \ell^4 - 2k^2\ell^2 + k^4}{2k^2\ell^2} \right] \sinh^2(2\ell a)$$

$$\frac{1}{T} = 1 + \left[\frac{\ell^4 + 2\ell^2k^2 + k^4}{2k^2\ell^2} \right] \sinh^2(2\ell a)$$

$$\frac{1}{T} = 1 + \left[\frac{(\ell^2 + k^2)^2}{2\ell^2k^2} \right] \sinh^2(2\ell a)$$

Use $k^2 = \frac{2mE}{\hbar^2}$ $\qquad \ell^2 = \frac{2m(V_0 - E)}{\hbar^2}$

$$\frac{1}{T} = 1 + \left[\frac{(V_0 - E + E)^2}{4E(V_0 - E)} \right] \sinh^2(2\ell a)$$

$$\frac{1}{T} = 1 + \frac{V_0^{\,2}}{4E(V_0 - E)} \sinh^2(2\ell a)$$

Let's investigate $E < V_0$, let $E = \frac{V_0}{2}$

$$\frac{1}{T} = 1 + \frac{(2E)^2}{4E(2E - E)} \sinh^2\left(\frac{2a}{\hbar}\sqrt{2m(2E-E)}\right)$$

$$= 1 + \sinh^2\left(\frac{2a}{\hbar}\sqrt{2mE}\right)$$

If $(\,) = 1$, $\frac{1}{T} = 1 + 1.4$, $T = 40\%$

$\Rightarrow$ 40% chance of making it through the barrier.

For an electron, how thick can the barrier be if $(\,) = 1$?

$$1 = \frac{2a}{\hbar}\sqrt{2mE}$$

$$E = \frac{\hbar^2}{8ma^2}$$

For an electron,

$$E = \frac{(1.05\times10^{-34}\,J\cdot s)^2}{8(9.11\times10^{-31}\,kg)} \cdot \frac{1}{d^2}$$

$$= \frac{1.5\times10^{-39}\,J\,m^2}{d^2}$$

So if we want to tunnel through a barrier of $1 \mathring{A}$ $= 10^{-10}m$ thickness, one atomic layer, the energy of the barrier should be

$$V_0 = 2E = 2\cdot\left(\frac{1.5\times10^{-39}\,J\cdot m^2}{(10^{-10}m)^2}\right)$$

$$= 3\times10^{-19}\,J = 1.8\,eV$$

with tunneling probability 40%

So a particle with energy $E = 0.9\,eV$ incident on a potential barrier $V_0 = 1.8\,eV$ of thickness $10^{-10}m$ will tunnel with probability 40%.