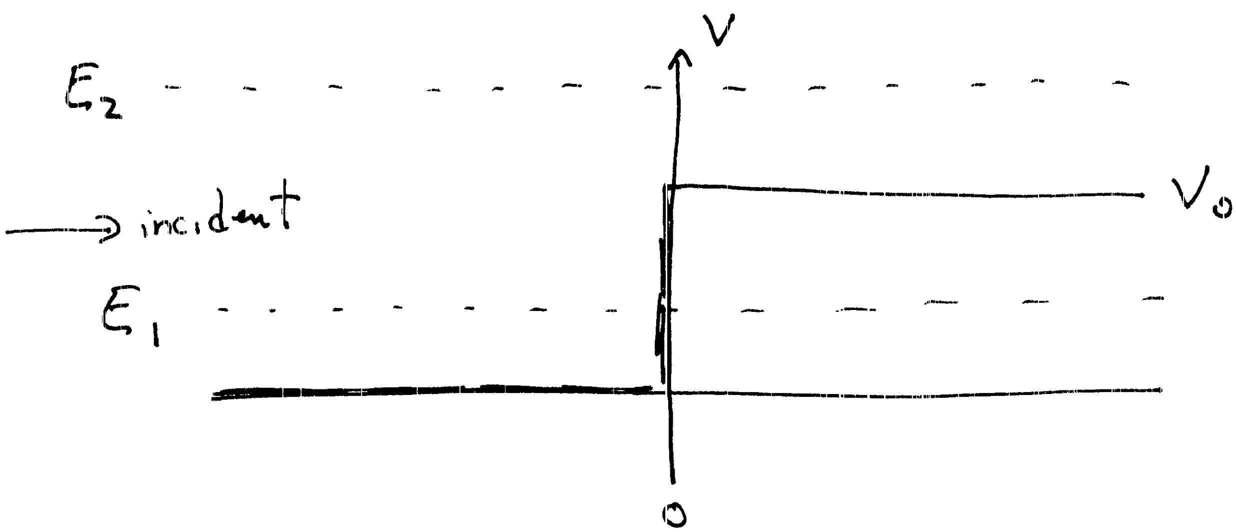
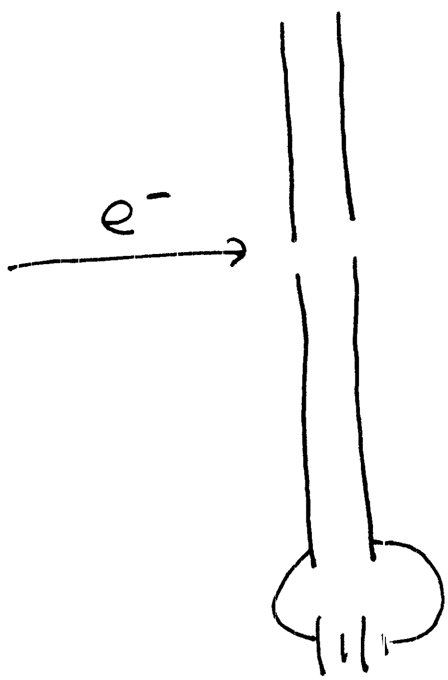
Step Potential



$$V(x) = \begin{cases} 0 & x < 0 \\ V_0 & x > 0 \end{cases}$$

Physically, how ~~a~~ would you make this ?

Parallel-plate capacitor



How do we think the particle will behave?

At E_1

Classically, the particle is reflected with 100% probability. There is no chance to find the particle at points $x>0$.

QM - The particle is still reflected with 100% probability; it can't travel an infinite distance with $E<U$. However, there is a non-zero chance to find the particle at $x>0$.

At E_2

CM, the particle is transmitted 100% of the time, but it slows for $x>0$.

QM, like a wave encountering a point where a string changes mass density, there is some ~~chan~~ chance the particle is transmitted and some chance it is reflected.

Consider $E_2 > V$

Solutions to TISE, with particle incident from left

$$\phi(x) = \begin{cases} \underbrace{A_I e^{ik_1 x}}_{\text{incident}} + \underbrace{A_{II} e^{-ik_1 x}}_{\text{reflected}} & x<0 \\ \underbrace{B_I e^{ik_2 x}}_{\text{transmitted}} & x>0 \end{cases}$$

* No negatively traveling wave if incident wave is from left.

For $x<0$, $\quad k_1 = \sqrt{\dfrac{2m(E-V)}{\hbar^2}} = \sqrt{\dfrac{2mE}{\hbar^2}}$

For $x>0$, $\quad k_2 = \sqrt{\dfrac{2m(E-V_0)}{\hbar^2}}$

Reflection and Transmission Coefficients

We calculated the probability current of a plane wave. If $\phi = Ae^{ikx}e^{-i\omega t}$

$$J = \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)$$

$$= \frac{i\hbar}{2m}AA^*\left(-ik - ik\right)$$

$$= AA^*\frac{\hbar k}{\cancel{2}m} = \text{density} \cdot v_{group}$$

To conserve particles at the interface

$$|J_{incident}| - |J_{reflected}| = |J_{transmitted}|$$

$$1 = \frac{|J_{reflected}|}{|J_{incident}|} + \frac{|J_{transmitted}|}{|J_{incident}|}$$

Define reflection coefficient R and transmission coefficient T

$$R = \frac{|J_{reflected}|}{|J_{incident}|}$$

$$T = \frac{|J_{transmitted}|}{|J_{incident}|}$$

In the case we are considering

$$R = \frac{A_{II}^* A_{II} (k_1 \hbar / m)}{A_I^* A_I (k_1 \hbar / m)} = \frac{A_{II}^* A_{II}}{A_I^* A_I}$$

$$T = \frac{B_I^* B_I (k_2 \hbar / m)}{A_I^* A_I (k_1 \hbar / m)} = \cancel{\frac{B_I^* k_2}{B_I}}$$

$$= \frac{B_I^* B_I \, k_2}{A_I^* A_I \, k_1}$$

Now solve for R, T. Note, since particles are conserved at the boundary

$$1 = R + T$$

so we can solve for R, then trivally compute T.

Impose Boundary Conditions

ϕ Continuous at $x=0$

$$A_I + A_{II} = B_I \qquad (1)$$

Derivative ϕ continuous at $x=0$

$$ik_1 A_I - ik_1 A_{II} = ik_2 B_I \qquad (2)$$

Solve for A_{II}/A_I

– Substitute (1) into (2)

$$k_1 A_I - k_1 A_{II} = k_2 (A_I + A_{II})$$

$$(k_1 - k_2) A_I = (k_1 + k_2) A_{II}$$

$$\frac{A_{II}}{A_I} = \frac{k_1 - k_2}{k_1 + k_2}$$

<u>Reflection Coefficient</u>

$$R = \frac{A_{II}^* A_{II}}{A_I^* A_I} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^* \left(\frac{k_1 - k_2}{k_1 + k_2}\right) = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2$$

<u>Transmission Coefficient</u>

$$T = 1 - R = 1 - \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2$$

$$= \frac{k_1^2 + 2k_1 k_2 + k_2^2 - (k_1^2 - 2k_1 k_2 + k_2^2)}{(k_1 + k_2)^2}$$

$$= \frac{4 k_1 k_2}{(k_1 + k_2)^2}$$

Solve for B_I

$$A_{II} = \frac{k_1 - k_2}{k_1 + k_2} A_I$$

$$A_I + A_{II} = B_I$$

$$A_I \left(1 + \frac{k_1 - k_2}{k_1 + k_2} \right) = B_I$$

$$A_I \left(\frac{2k_1}{k_1 + k_2} \right) = B_I$$

$$\frac{B_I}{A_I} = \frac{2k_1}{k_1 + k_2}$$

Note, $T \neq B_I^* B_I / A_I^* A_I$

If $V_1 < V_2$, $k_1 > k_2$ and A_{II} is positive

$\Rightarrow$ incident and reflected wave in phase

If $V_1 > V_2$, $k_1 < k_2$ and A_{II} is negative

$\Rightarrow$ ~~A_{II}~~ $A_{II} = e^{i\pi} |A_{II}|$

$\Rightarrow$ reflected and incident wave are 180°
out of phase.

Now consider E, $E < V$

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}} \quad \text{is unchanged}$$

$$k_2 = \sqrt{\frac{2m(E-V)}{\hbar^2}} = \sqrt{-1}\sqrt{\frac{2m(V-E)}{\hbar^2}}$$

$$= i\kappa_2$$

$$\kappa_2 = \sqrt{\frac{2m(V-E)}{\hbar^2}} \quad \text{real}$$

At no point did I assume k_2 was real,

$$\frac{A_{II}}{A_I} = \frac{k_1 - k_2}{k_1 + k_2} = \frac{k_1 - ik_2}{k_1 + ik_2}$$

$$\frac{B_I}{\cancel{A}B_{II}} = \frac{2k_1}{k_1 + k_2} = \frac{2k_1}{k_1 + ik_2}$$

Compute R, T

$$R = \frac{A_{II}^* A_{II}}{A_I^* A_I} = \left(\frac{k_1 + ik_2}{k_1 - ik_2}\right)\left(\frac{k_1 - ik_2}{k_1 + ik_2}\right) = 1$$

$$T = 1 - R = 0$$

If $E < V$, all the wave is eventually reflected