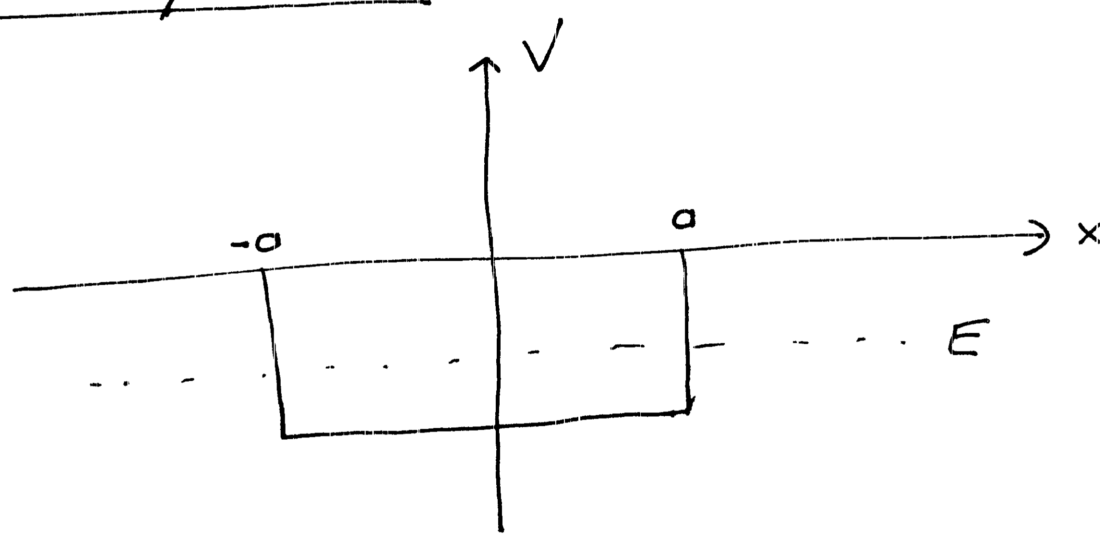
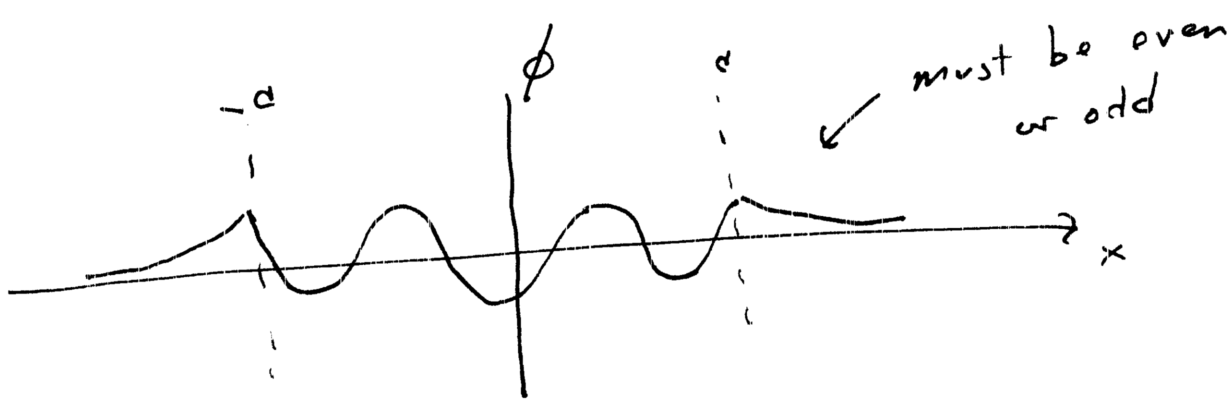
Finite Square Well

Thm – If the well is symmetric about $x=0$, the probability density $\rho(x) = \phi^* \phi$ must be symmetric about $x=0$, therefore $\phi(x)$ must be, either an even or odd function for bound states.

Finite Square Well





Solution TISE

$$\phi(x) = \begin{cases} A e^{\ell x} & x < -a \\ B e^{ikx} + C e^{-ikx} & -a < x < a \\ D e^{-\ell a} \end{cases}$$

discarding solutions that go to ∞.

However, $\phi(x)$ must be even or odd
Even $\phi(-x) = \phi(x)$, odd $\phi(-x) = -\phi(x)$

Even

$$\phi(x) = \begin{cases} A e^{\ell x} & x < -a \\ B \cos kx & -a < x < a \\ A e^{-\ell x} & x > a \end{cases}$$

Odd

$$\phi(x) = \begin{cases} -A e^{\ell x} & x < -a \\ B \sin kx & -a < x < a \\ A e^{-\ell x} & x > a \end{cases}$$

What are we looking for?

(1) R, T do not make sense if $E < V$.

(2) Energy

(3) Wave function up to a constant that can be determined by normalization.

Let $V = -V_0$, $V_0 > 0$ and $E < 0$

but $|E| \leq |V_0| \Rightarrow E + V_0 > 0$

Outside the well, $V = 0$

$$\ell = \sqrt{\frac{-2mE}{\hbar^2}}$$

+ from $V = -V_0$

Inside the well

$$k = \sqrt{\frac{2m(E+V_0)}{\hbar^2}} \quad \text{real}$$

Suspect Ground State is even, so work on the even case.

Even

$$\phi(x) = \begin{cases} Ae^{\ell x} & x < -a \\ B\cos kx & -a < x < a \\ Ae^{-\ell x} & x > a \end{cases}$$

Boundary Conditions

(1) ϕ continuous at $x = \pm a$

At $x = a$,

$$B\cos ka = Ae^{-\ell a} \qquad (1)$$

(2) Slope $\frac{d\phi}{dx}$ continuous

$$-Bk\sin ka = -\ell Ae^{-\ell a} \qquad (2)$$

Divide 2/1

$$k \tan(ka) = \ell \qquad (3)$$

We want to solve (3) for the energy, but both k and ℓ depend on E, so first we have to write one in terms of the other.

Define the dimensionless quantity

$$z = ka = a\sqrt{\frac{2m(E+V_0)}{\hbar^2}}$$

$$z^2 = \frac{a^2}{\hbar^2} 2m(E+V_0)$$

$$= \frac{2ma^2E}{\hbar^2} + \frac{2ma^2V_0}{\hbar^2}$$

Define $z_0^2 = \dfrac{2ma^2V_0}{\hbar^2}$

$$z^2 = a^2 \left(\frac{2mE}{\hbar^2} \right) + z_0^2$$

but $$\ell^2 = \frac{-2mE}{\hbar^2}$$

so

$$z^2 = -\ell^2 a^2 + z_0^2$$

$$\ell^2 a^2 = z_0^2 - z^2$$

Back to our original equation

$$k \tan(ka) = \ell$$

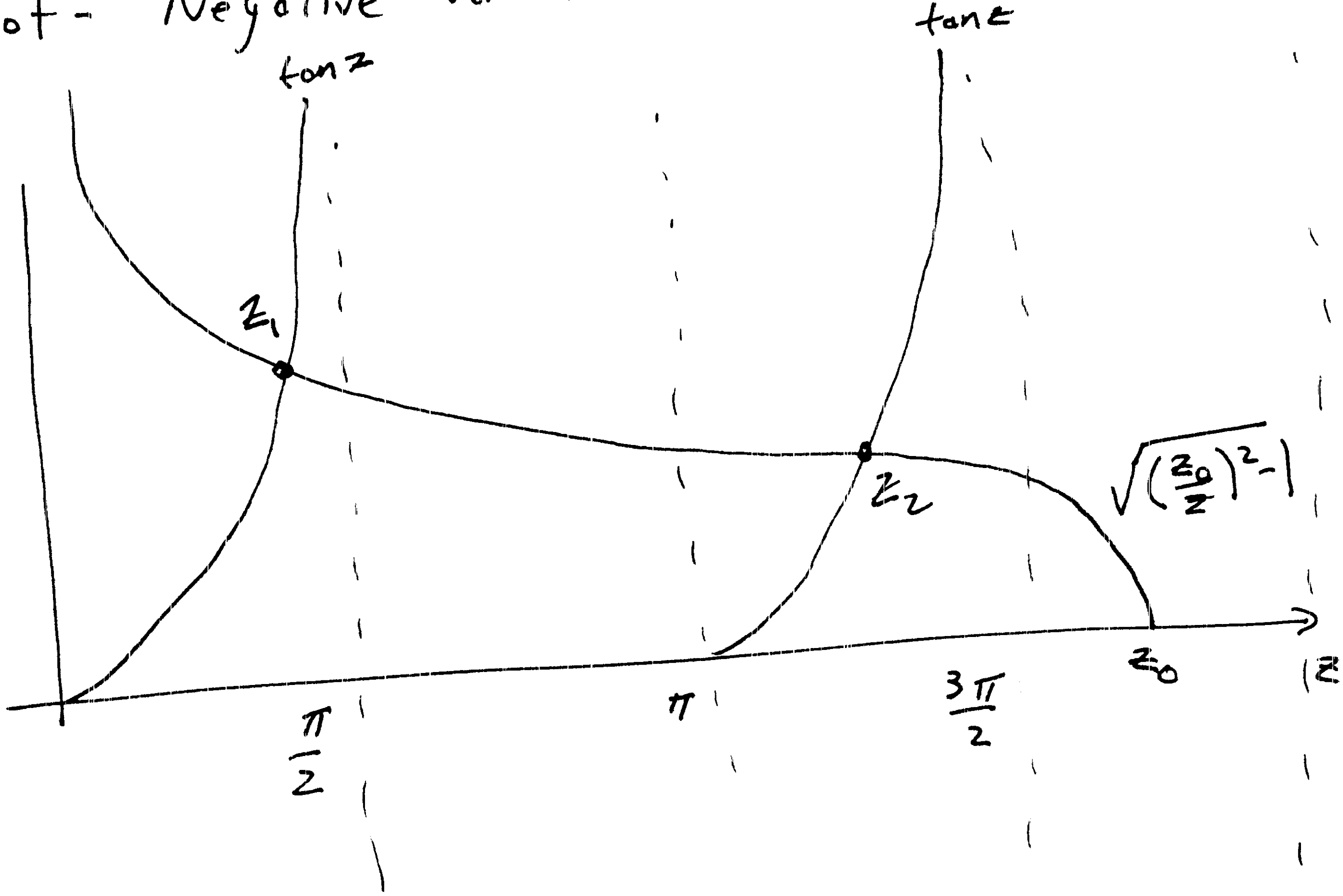
$$\frac{z}{a} \tan(z) = \sqrt{\frac{z_0^2 - z^2}{a^2}}$$

$$z \tan(z) = \sqrt{z_0^2 - z^2}$$

$$\sqrt{\left(\frac{z_0}{z}\right)^2 - 1} = \tan(z)$$

which has no closed form solution.

Plot - Negative values of $\tan(z)$ cannot be solutions



So z_0 controls the number of solutions. If $z_0 < \pi$ there is one solution, one energy. If $\pi < z_0 < 2\pi$ there are two solutions, two steady state energies in the well.

How do we find the energy?

1. Compute $z_0^2 = \dfrac{2ma^2 V_0}{\hbar^2}$ for our potential well and particle.

2. Use a numeric solver to find z, for example in Maple

$$\text{fsolve}\left(\sqrt{\left(\frac{z_0}{z}\right)^2 - 1} = \tan(z) \, , \, z \right)$$

yields $z = 1.14$ if $z_0 = \dfrac{7\pi}{4}$.

3. From z,

$$\cancel{z^2 = k^2 a^2} = \frac{2m(E+V_0)}{\hbar^2} a^2$$

$$z^2 = k^2 a^2 =$$

$$\Rightarrow E = \frac{\hbar^2 z^2}{2ma^2} - V_0$$

Odd States

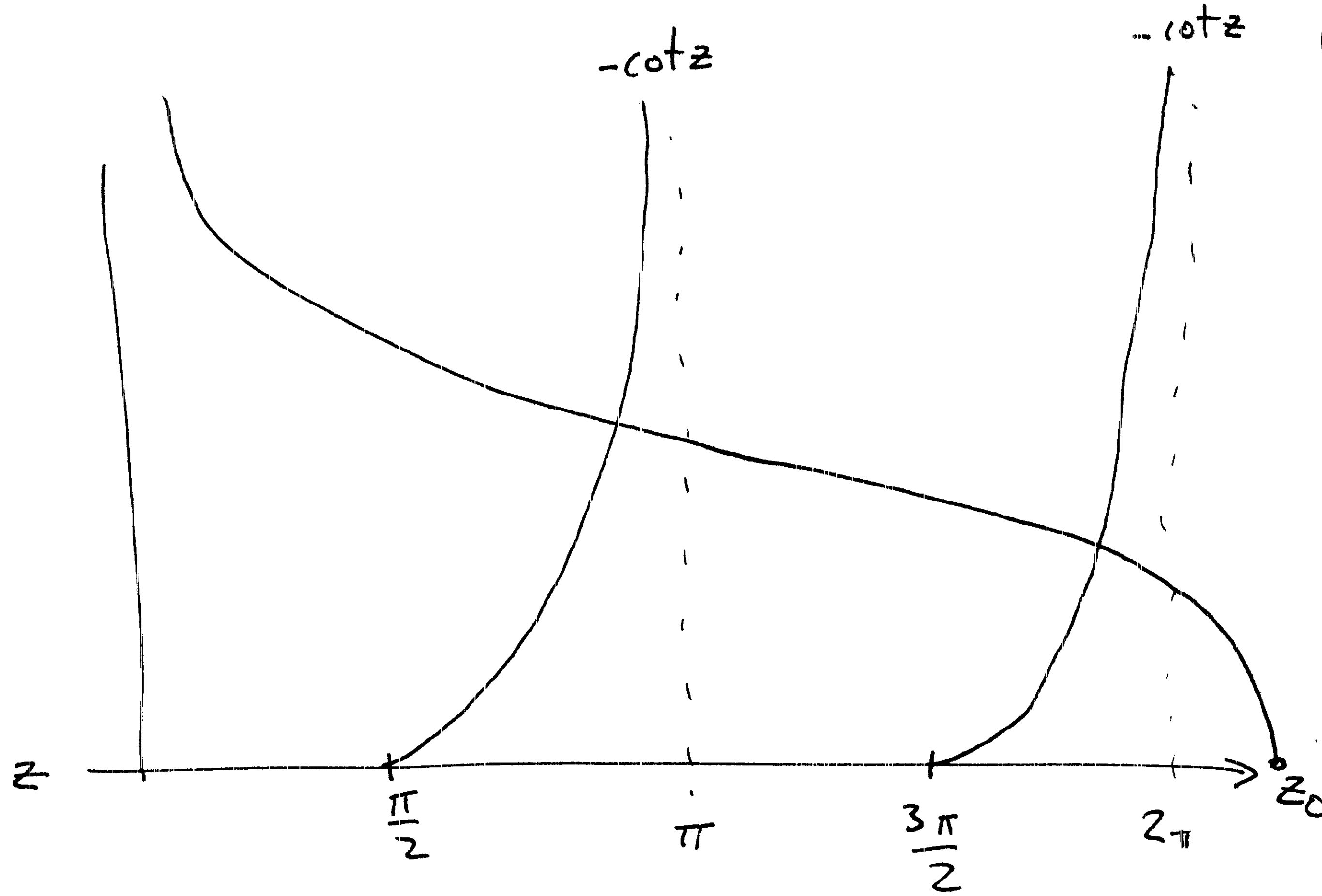
$$\phi(x) = \begin{cases} -A e^{+\ell x} & x < -a \\ B \sin kx & -a < x < a \\ A e^{-\ell x} & x > a \end{cases}$$

Same derivation yields eqn

$$-\ell = k \cot(ka)$$

or

$$-\cot(z) = \sqrt{\left(\frac{z_0}{z}\right)^2 - 1}$$



If $z_0 < \frac{\pi}{2}$, no odd solutions.

$\Longrightarrow$ The finite square well must have at least one even bound state, but <u>MAY</u> have no odd bound states.