The Free Particle

For a free particle $V=0$, and the TISE is

$$\frac{-\hbar^2}{2m}\frac{d^2\phi}{dx^2} = E\phi$$

$$\frac{d^2\phi}{dx^2} + \frac{2mE}{\hbar^2}\phi = 0$$

$$\frac{d^2\phi}{dx^2} + k^2\phi = 0$$

$$\phi = \frac{1}{\sqrt{2\pi\hbar}} e^{\pm ikx}$$ (traditional normalization)

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} e^{\pm ikx - i\omega t}$$

Dispersion Relation

$$E = \hbar\omega = \frac{\hbar^2 k^2}{2m}$$

<u>Phase Velocity</u> - Speed of a point of constant phase $\phi = \pm kx - \omega t$

$$d\phi = 0 = \pm k\,dx - \omega\,dt$$

$$\frac{dx}{dt} = \pm \frac{\omega}{k}$$

$\Longrightarrow$ If $k>0$, wave travels to $+x$

$k<0$ " " " $-x$

We already showed the wave

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}}\, e^{ikx-\omega t}$$

has momentum wave function

$$\Phi(p) = \delta(p-p')$$

and the wave has momentum of exactly

$$p = \hbar k$$

This means the wave's speed should be $v_{classical} = \frac{\hbar k}{m}$

We can calculate the wave's phase velocity from the dispersion relation

$$\frac{dx}{dt} = \frac{\omega}{k} = \frac{1}{k}\frac{\hbar k^2}{2m}$$

$$= \frac{\hbar k}{2m} = \frac{v_{classical}}{2}$$

$\Rightarrow$ The QM plane wave travels half as fast as it should.

To investigate this, on we know something's wrong since $\frac{d\langle x\rangle}{dt} = \langle p\rangle/m$

we need to look at wave packets.

A wave packet will mix a bunch of waves together to give a normalizable wave function

$$\phi(x) = \sum_n c_n e^{ik_n x}$$

This is the same thing we wrote for the infinite square well, except the solutions to the TISE were $\phi_i = \sqrt{\frac{2}{L}} \sin k_n x$

There is a problem though. Nothing the free particle suggests that $E(k)$ or k is discrete so we should really write

$$\phi(x) = \sum_k c(k) e^{ikx}$$

OR

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int \bar{\phi}(k)\, e^{ikx}\, dk$$

traditional normalization

Note, this is just the Fourier transform and we see $\bar{\phi}(k)$ plays the role of $c(k)$ the coefficients of the expansion.

Fourier transforms can be inverted

$$\bar{\Phi}(k) = \frac{1}{\sqrt{2\pi}} \int \phi(x) e^{-ikx} dx$$

This looks aalot like the momentum wave function definition. Does a similar probabalistic interpretation exist?

What is $\rho(k)$, the probability density for the observation of k?

$$P(k \in [k_1, k_2]) = \int_{k_1}^{k_2} \rho(k)\, dk$$ probability $k \in [k_1, k_2]$

$$= \int_{p_1}^{p_2} \rho(p)\, dp = \int_{\hbar k_1}^{\hbar k_2} \rho(p)\, dp$$

Let $p = \hbar k$, $dp = \hbar dk$

$$P(k \in [k_1, k_2]) = \int_{k_1}^{k_2} \hbar\, \rho(\hbar k)\, dk$$

$$= \int_{k_1}^{k_2} (\sqrt{\hbar}\, \bar{\Phi}(p))^* (\sqrt{\hbar}\, \bar{\Phi}(p))\, dk$$

So we get the desired probabalistic definition of $\bar{\Phi}(k)$ as $P(k) = \bar{\Phi}^*(k)\bar{\Phi}(k)$

if $\bar{\Phi}(k) = \sqrt{\hbar}\,\Phi(p=\hbar k)$ ← momentum wave function

$$= \frac{\sqrt{\hbar}}{\sqrt{2\pi\hbar}} \int \phi(x)\, e^{-i\hbar kx/\hbar}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int \phi(x)\, e^{-ikx}\, dx$$

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Time Evolution

If everything was discrete and if we knew the expansion of the initial state $\phi(x)$ in terms of the solutions to the TISE ϕ_i

$$\phi(x) = \sum c_i \phi_i(x)$$

we could immediately write the time evolution

$$\psi(x,t) = \sum c_i \phi_i(x)\, e^{-i\omega_i t}$$

$$E_i = \hbar\omega_i$$

We can do the same for the free particle

$$E = \hbar\omega = \frac{\hbar^2 k^2}{2m}$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int \phi(k)\, e^{ikx}\, dk$$

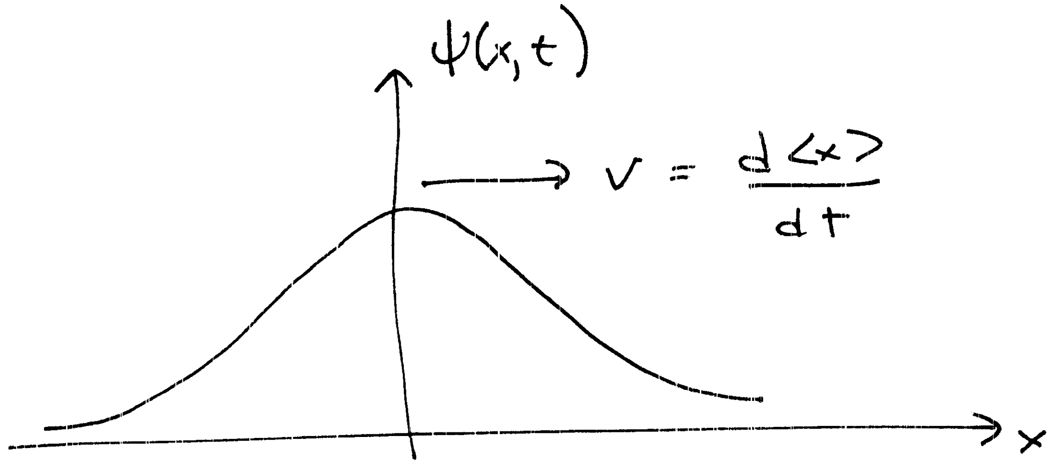
$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int \phi(k)\, e^{ikx}\, e^{-i\omega_k t}\, dk$$

$$\omega_k = \frac{\hbar k^2}{2m}$$

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int \phi(k)\, e^{i\left[kx - \frac{\hbar k^2 t}{2m}\right]}\, dk$$

This expression provides the time evolution of any initial free particle wave function $\psi(x,0) = \phi(x)$

Wave Packet – A localized disturbance formed of many plane waves



The speed of the average location of the wave $\langle x\rangle$ is called the group velocity,

$$v_{group} = \frac{d\langle x\rangle}{dt} = \frac{d\omega}{dk}$$

$\Rightarrow$ Each wave forming the packet still moves at $v_{phase} = \omega/k$.

$\Rightarrow$ For plane waves, $\omega = \frac{\hbar k^2}{2m}$

$$\frac{d\omega}{dk} = \frac{\hbar k}{m} = \frac{p}{m} = v_{classical}$$