Gaussian Integrals

Many quantities have probability distributions that are well approximated by the normal distribution which has a Gaussian probability density

$$p(x) = A e^{-ax^2}$$

where A and a are constants.

To be a probability,

$$1 = \int_{-\infty}^{\infty} A e^{-ax^2} dx$$

Let $I = \int_{-\infty}^{\infty} e^{-ax^2} dx$

$$I^2 = \int_{-\infty}^{\infty} e^{-ax^2} dx \int_{-\infty}^{\infty} e^{-ay^2} dy$$

$$= \int_0^{\infty} dr \int_0^{2\pi} r d\theta \, e^{-ar^2} \qquad r^2 = x^2 + y^2$$

$\Rightarrow$ Change to polar coordinates

Perform θ integral

$$I^2 = 2\pi \int_0^\infty r e^{-ar^2} dr$$

u-substitution

$$u = -ar^2 \qquad du = -2ar\,dr$$

$$I^2 = -\frac{\pi}{a} \int_0^{-\infty} e^u du$$

$$= -\frac{\pi}{a} e^u \Big|_0^{-\infty}$$

$$= \frac{\pi}{a}$$

$$I = \sqrt{\frac{\pi}{a}} = \int_{-\infty}^{\infty} e^{-ax^2} dx$$

$$1 = A \int_{-\infty}^{\infty} e^{-ax^2} dx = A\sqrt{\frac{\pi}{a}}$$

$$A = \sqrt{\frac{a}{\pi}}$$

$$\boxed{p(x) = \sqrt{\frac{a}{\pi}}\, e^{-ax^2}}$$

We can also calculate the standard deviation of the Gaussian probability density

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

The probability density is peaked at $x=0$ so the average is $\langle x \rangle = 0$.

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 p(x)\, dx$$

$$= \sqrt{\frac{a}{\pi}} \int_{-\infty}^{\infty} x^2 e^{-ax^2}\, dx$$

$$= 2\sqrt{\frac{a}{\pi}} \int_0^{\infty} x^2 e^{-ax^2}\, dx$$

$$\cancel{= 2\sqrt{\frac{a}{\pi}} \frac{\sqrt{\pi}}{a^{3/2}}}$$

$$= \left(2\sqrt{\frac{a}{\pi}}\right) \frac{1}{4} \frac{\sqrt{\pi}}{a^{3/2}}$$

$$= \frac{1}{2a}$$

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

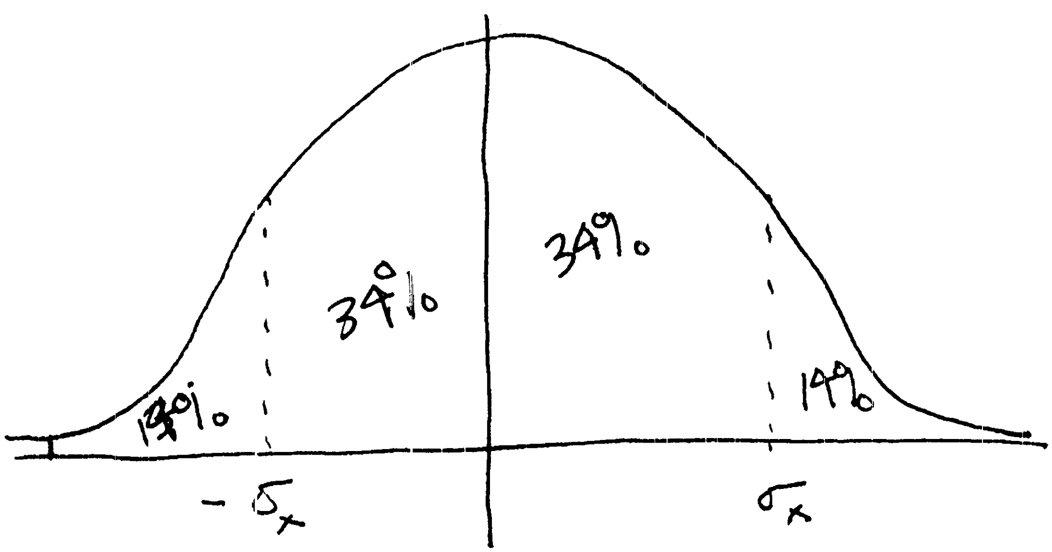
$$= \sqrt{\frac{1}{2a}}$$

$$\sigma_x^2 = \frac{1}{2a}$$

$$a = \frac{1}{2\sigma_x^2}$$

So we can re-write the probability density as

$$p(x) = \frac{1}{\sigma_x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_x^2}}$$



Integration by Parts

Finding $\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx$

$$d uv = u\,dv + v\,du$$

We want to remove one power of x from x^2

$$u = x \qquad dv = x e^{-ax^2} dx$$

$$\int_{-\infty}^{\infty} u\,dv = \int_{-\infty}^{\infty} d(uv) - \int_{-\infty}^{\infty} v\,du$$

$$v = \int x e^{-ax^2} dx$$

$$= -\frac{1}{2}\frac{e^{-ax^2}}{a}$$

$$du = dx$$

$$\int_{-\infty}^{\infty} u\,dv = -\frac{1}{2a} x e^{-ax^2}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\left(-\frac{1}{2a}e^{-ax^2}\right)dx$$

The first term is zero.

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{1}{2a} \int_{-\infty}^{\infty} e^{-ax^2} dx$$

$$= \frac{1}{2a} \sqrt{\frac{\pi}{a}}$$

$$= \frac{\sqrt{\pi}}{2a^{3/2}}$$

> $integrate\left(exp\left(-a\cdot x^2\right),\ x=-infinity..infinity\right);$

$$\begin{cases} \dfrac{\sqrt{\pi}}{\sqrt{a}} & \mathrm{csgn}(a)=1 \\ \infty & otherwise \end{cases} \tag{1}$$

> $integrate\left(\left(\dfrac{a}{Pi}\right)^{\left(\frac{1}{2}\right)}\cdot x^2\cdot exp\left(-a\cdot x^2\right),\ x=-infinity..infinity\right);$

$$\begin{cases} \dfrac{1}{2}\,\dfrac{1}{a} & \mathrm{csgn}(a)=1 \\ \infty & otherwise \end{cases} \tag{2}$$

> $integrate\left(x^2\cdot exp\left(-a\cdot x^2\right),\ x=-infinity..\infty\right);$

$$\begin{cases} \dfrac{1}{2}\,\dfrac{\sqrt{\pi}}{a^{(3/2)}} & \mathrm{csgn}(a)=1 \\ \infty & otherwise \end{cases} \tag{3}$$

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