PHYS 4073 - Quantum Mechanics- Homework Set 1

Reading Assignment: Chapter 1

Due 5:45pm Monday August 30th in my box in the physics office or in office hours.

Griffiths' Problems

1.1

1.3

Additional Problems

Problem A1 - Complex Numbers A little practice with complex numbers.

(a) Write $1/(1+i)$ in the form $a+bi$ and the form $\rho e^{i\theta}$

(b) Write $2e^{i\pi/6}$ in the form $a+ib$

(c) Compute $\sqrt{1+2i}$

Problem A2 - Complex Numbers, But More Interesting Use complex numbers to compute a more expressive form of the sum of two sine waves of different frequency ω and 3ω but equal amplitude A. Show the sum of the two waves $f = A\sin\omega t + A\sin 3\omega t$ can be written as the product of a wave with a frequency that is the average of the two frequencies multiplied by a wave whose frequency is the difference of the two frequencies. Start by writing f as the imaginary part of some complex exponentials.

Problem A3 - Wave Functions Consider the ground state of the infinite square well, $\psi(x) = A\sin(\pi x/a)$ for $0 < x < a$ and $\psi(x) = 0$ otherwise.

(a) Find A.

(b) Find $< x >$.

(c) Find σ_x.

(d) Find the probability the particle is found in the middle of the well from $a/4$ to $3a/4$.

Problem A4 - Double Slit Monochromatic (single frequency, ω) light is incident on two slits. The slits are a distance d apart, are very narrow, and are a distance ℓ from a screen. The electric field of the wave is given by $E(x,t) = E_0\sin(kx - \omega t)$ where k is the wavenumber $k = 2\pi/\lambda$ and λ is the wavelength. The intensity of the light on the screen in naturally proportional to E^2. Work out the location of the minima of the interference pattern as measured by the distance y from the central maximum. You may assume $d << \ell$ and naturally $c = \omega/k$, but that's not important.

Problem A5 - Baseballs The quantum (De Broglie) wavelength of anything is $\lambda = h/mv$ where m is the mass, v is the velocity, and h is Planck's constant. Compute the spacing of the central maximum and the first interference minima for a baseball thrown at 100mph and a wall containing two windows a distance 5ft apart. The interference pattern is observed on a wall a distance of 20ft from the windows.

Problem A6 - From the GRE Consider a stepwise wave function $\psi = 0$ for $x < 0$, $\psi = 1$ for $0 < x < 1$, $\psi = 2$ for $1 < x < 2$, $\psi = 5$ for $2 < x < 3$, $\psi = 1$ for $3 < x < 4$, and $\psi = 0$ for $x > 4$. Find the probability the particle is found between $x = 2$ and $x = 3$. Naturally, you are not going to consider working with a non-normalized wave function.

1.1

Count	Age	$P(j)$	$j - \langle j \rangle$
1	14	1/14	-7
1	15	1/14	-6
3	16	3/14	-5
2	22	2/14	1
2	24	2/14	3
5	25	5/14	4
Total 14			

(a)

$$\langle j \rangle = \sum j P(j)$$

$$= 14 \cdot \frac{1}{14} + 15 \cdot \frac{1}{14} + 16 \cdot \frac{3}{14}$$

$$+ 22 \cdot \frac{2}{14} + 24 \cdot \frac{2}{14} + 25 \cdot 5/14$$

$$= 21$$

$$\langle j \rangle^2 = 441$$

$$\langle j^2 \rangle = \sum j^2 P(j)$$

$$= 14^2 \cdot \frac{1}{14} + 15^2 \cdot \frac{1}{14} + 16^2 \cdot \frac{3}{14}$$

$$+ 22^2 \cdot \frac{2}{14} + 24^2 \cdot \frac{2}{14} + 25^2 \cdot \frac{5}{14}$$

$$= 459.57$$

(b) See table above for $j - \langle j \rangle = \Delta j$

$$\langle (\Delta j)^2 \rangle = \sum \Delta j^2 P(j)$$

$$= 7^2 \frac{1}{14} + 6^2 \frac{1}{14} + 5^2 \frac{3}{14}$$

$$+ 1^2 \frac{2}{14} + 3^2 \frac{2}{14} + 4^2 \frac{5}{14}$$

$$= 18.57 \qquad \sigma = \sqrt{18.57} = 4.31$$

(c) $\sigma = \sqrt{\langle j^2 \rangle - \langle j \rangle^2}$

$$= \sqrt{\cancel{441}\ 459.57 - 441} = 4.31$$

1.3

(a)

$$1 = \int_{-\infty}^{\infty} A e^{-\lambda(x-a)^2} dx$$

$$= \int_{-\infty}^{\infty} A e^{-\lambda x^2} dx$$

using $u = x - a$

From lecture

$$\int_{-\infty}^{\infty} e^{-\lambda x^2} dx = \sqrt{\frac{\pi}{\lambda}}$$

$$A = \sqrt{\frac{\lambda}{\pi}}$$

(b)

$$\langle x\rangle = \sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty} x\, e^{-\lambda(x-a)^2}\, dx$$

$$= \sqrt{\frac{\lambda}{\pi}}\int (u+a)\, e^{-\lambda u^2}\, du \qquad u = x-a$$

$$= \underbrace{a\sqrt{\frac{\lambda}{\pi}}\int e^{-\lambda u^2} du}_{a} + \underbrace{\sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty} u e^{-\lambda u^2} du}_{0}$$

$\langle x\rangle = a$ (as it had to be)

$$\langle x^2\rangle = \sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty} x^2 e^{-\lambda(x-a)^2} dx$$

$$= \sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty} (u+a)^2 e^{-\lambda u^2} du \qquad u = x-a$$

$$= \sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty} u^2 e^{-\lambda u^2} du + 2a\sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty} u\, e^{-\lambda u^2} du$$

$$+ a^2\sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty} e^{-\lambda u^2} du$$

The first integral was done in lecture and is equal to $1/2\lambda$. The second integral is zero and the third is a^2

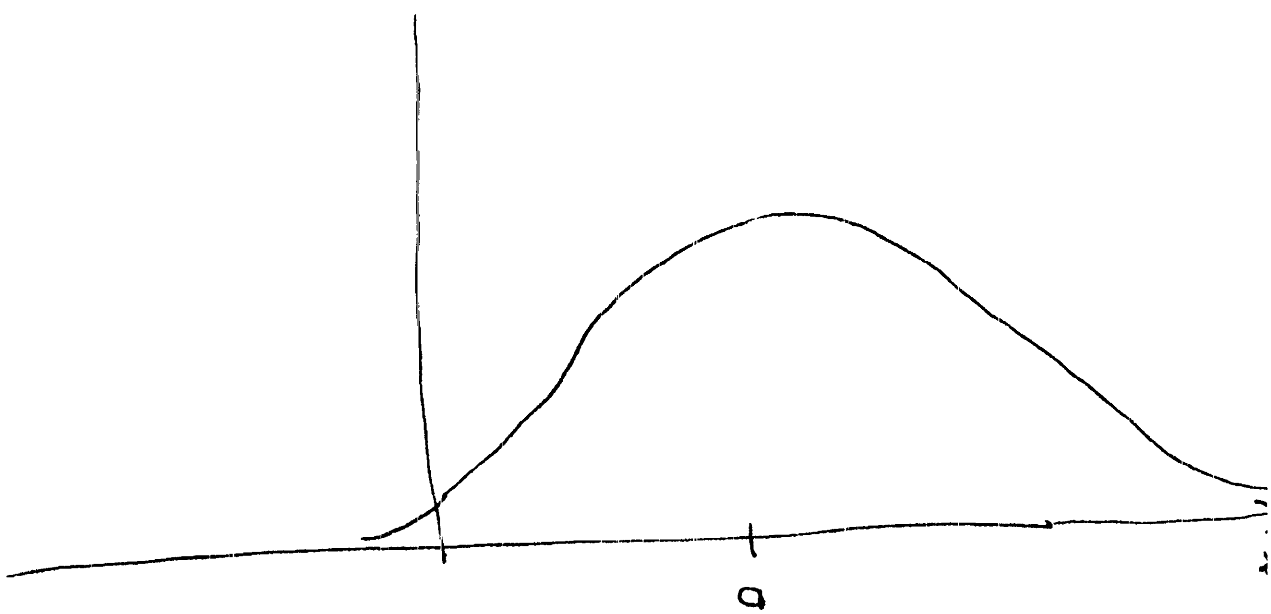
$$\langle x^2 \rangle = \frac{1}{2\lambda} + a^2$$

$$\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$= \sqrt{\frac{1}{2\lambda} + a^2 - a^2}$$

$$\sigma = \frac{1}{\sqrt{2\lambda}}$$

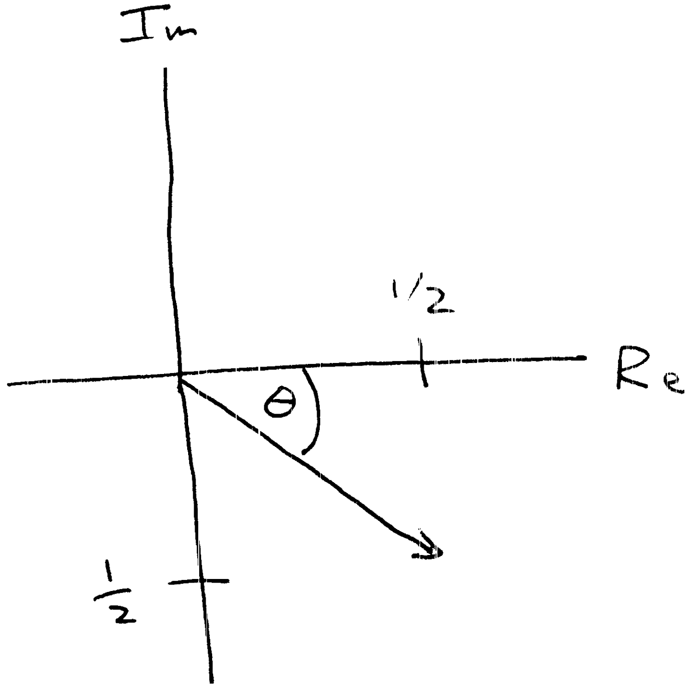
(c)



(A1) (a) Multiply by complex conjugate

$$z = \frac{1}{1+i} = \frac{1}{1+i} \cdot \frac{1-i}{1-i} = \frac{1-i}{2}$$

$$= \frac{1}{2} - \frac{1}{2} i$$



$$\tan\theta = \frac{-1/2}{1/2} = -1 \qquad \theta = -\frac{\pi}{4}$$

$$\rho = \sqrt{\frac{1}{2}^2 + \frac{1}{2}^2} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$z = \frac{\sqrt{2}}{2} e^{-i\pi/4}$$

(b) $\sin\frac{\pi}{6} = \frac{1}{2}$ $\qquad$ $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

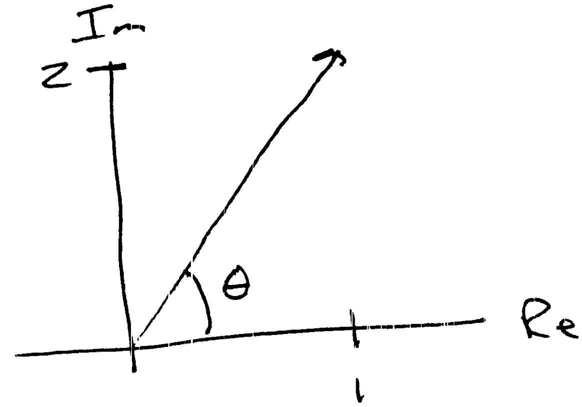
$$z = 2e^{i\pi/6} = 2\cos\pi/6 + 2i\sin\pi/6$$

$$= 2\cdot\frac{\sqrt{3}}{2} + 2i\cdot\frac{1}{2} = \sqrt{3} + i$$

(c) $z = \sqrt{1+2i}$

Write $1+2i$ as $\rho e^{i\theta}$

$$\rho = \sqrt{1^2+2^2} = \sqrt{5}$$



$$\tan\theta = \frac{2}{1} \Rightarrow \theta = 63.4^\circ$$

$$\sqrt{z} = \sqrt{\rho}\, e^{i\theta/2} = 5^{1/4} e^{i\,31.7^\circ}$$

$$= 1.27 + 0.79i$$

A2

$$f = A\sin\omega t + A\sin 3\omega t$$

$$= \mathrm{Im}\left(Ae^{i\omega t} + Ae^{3i\omega t}\right)$$

$$= \mathrm{Im}\left(Ae^{2i\omega t}\left(e^{-i\omega t} + e^{i\omega t}\right)\right)$$

$$e^{-i\omega t} + e^{i\omega t} = \left(\cos\omega t - i\sin\omega t + \cos\omega t + i\sin\omega t\right)$$

$$= 2\cos\omega t$$

Can bring real stuff out of Im

$$f = A\,\mathrm{Im}\left(2\cos\omega t\, e^{2i\omega t}\right)$$

$$= 2A\cos\omega t\,\mathrm{Im}\, e^{2i\omega t} = 2A\cos\omega t\sin 2\omega t$$

$\omega = \dfrac{3\omega - \omega}{2}$ = $\frac{1}{2}$ Difference of frequencies

$2\omega = \dfrac{\omega + 3\omega}{2}$ = Average of frequencies

A3

(d) Normalize

$$1 = \int_0^a \psi^* \psi \, dx = AA^* \int_0^a \sin^2 \frac{\pi x}{a} \, dx$$

<u>Wolfram Alpha</u>

$$\int \sin^2 \frac{\pi x}{a} \, dx = \frac{x}{2} - \frac{a \sin\left(\frac{2\pi x}{a}\right)}{4\pi} + C$$

$$\int_0^a \sin^2 \frac{\pi x}{a} \, dx = \frac{x}{2} - \frac{a}{4\pi} \sin\left(\frac{2\pi x}{a}\right) \Bigg|_0^a$$

$$= \frac{a}{2}$$

$$1 = AA^* \frac{a}{2}$$

$$A = \sqrt{\frac{2}{a}}$$

Alternately $$\int_0^a \sin^2 \frac{\pi x}{a}\, dx = \int_0^a \cos^2 \frac{\pi x}{a}\, dx$$

$$\int_0^a \sin^2 \frac{\pi x}{a}\, dx = \frac{1}{2} \int_0^a \left(\sin^2 \frac{\pi x}{2} + \cos^2 \frac{\pi x}{2} \right) dx$$

$$= \frac{1}{2} \int_0^a dx = \frac{a}{2}$$

(b) Expected Value

$$\langle x \rangle \equiv \int_0^a x\, \psi^* \psi\, dx = \frac{2}{a} \int_0^a x \sin^2 \frac{\pi x}{a}\, dx$$

Wolfrom Alpha

$$\int_0^a x \sin^2 \frac{\pi x}{a}\, dx = \frac{a^2}{4}$$

$$\langle x \rangle = \frac{2}{a} \cdot \frac{a^2}{4} = \frac{a}{2}$$

which is what it had to be.

(c) $\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$

$$\langle x^2\rangle = \int x^2 \psi^* \psi \, dx = \frac{2}{a}\int_0^a x^2 \sin^2 \frac{\pi x}{a}\, dx$$

$$= \frac{2}{a} \cdot \frac{1}{12}\left(2 - \frac{3}{\pi^2}\right) a^3$$

$$= \frac{a^2}{3} - \frac{1}{2}\frac{a^2}{\pi^2}$$

Note units check $\langle x^2\rangle$ must have units length^2 and a^2 has units length^2.

$$\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2 = \frac{a^2}{3} - \frac{1}{2}\frac{a^2}{\pi^2} - \frac{a^2}{4}$$

$$= a^2\left(\frac{1}{12} - \frac{1}{2\pi^2}\right) > 0$$

(d) $\mathbb{P}\left(x \in \left[\frac{a}{4}, \frac{3a}{4}\right]\right)$

$$= \int_{a/4}^{3a/4} \psi^* \psi \, dx = \frac{2}{a} \int_{a/4}^{3a/4} \sin^2 \frac{\pi x}{a} \, dx$$

$$= \left. \frac{x}{2} - \frac{a}{4\pi} \sin\left(\frac{2\pi x}{a}\right) \right|_{a/4}^{3a/4}$$

$$= \frac{2}{a}\left[\frac{1}{2}\left(\frac{3a}{4} - \frac{a}{4}\right) - \frac{a}{4\pi}\underbrace{\left(\underset{\substack{\shortparallel \\ -1}}{\sin\frac{3}{2}\pi} - \underset{\substack{\, \\ 1}}{\sin\frac{\pi}{2}}\right)}_{-2}\right]$$

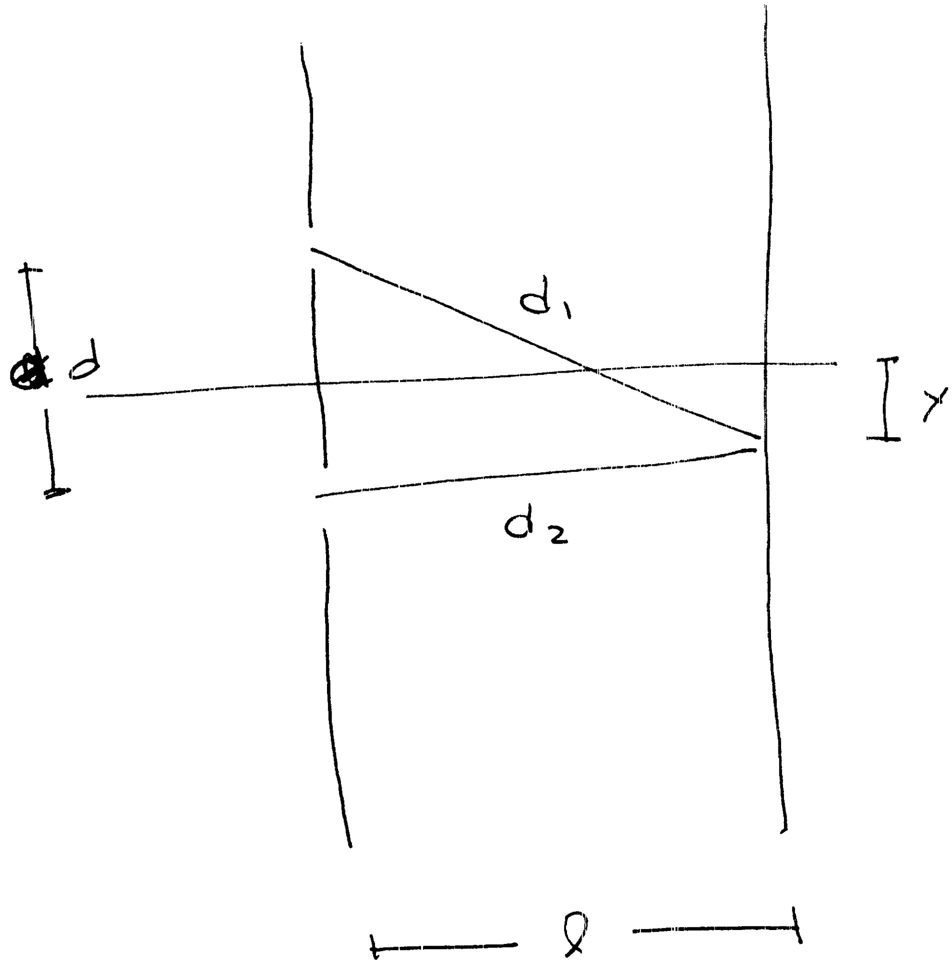
$$= \left[\frac{1}{8} \cdot 2a + \frac{2a}{4\pi}\right] \cdot \frac{2}{a}$$

$$= \frac{2}{a} \cdot \left(\frac{a}{4} + \frac{a}{2\pi}\right)$$

$$= \frac{1}{2} + \frac{1}{\pi} = 0.81$$

Note, less than 1 because prob.

(A4)



The intensity is

$$I \propto E^2 = E_0^2 (\sin kd_1 + \sin kd_2)^2$$

The term in ( ) is zero if

$$kd_1 - kd_2 = n\pi$$

that is if the waves are 180° out of phase

Write out d_1, d_2

$$d_1 = \sqrt{\ell^2 + (y + d/2)^2}$$

$$d_2 = \sqrt{\ell^2 + (y - d/2)^2}$$

Binomial Expansion - But first we have to collect a small parameter.

$$d_1 = \ell\sqrt{1 + \left(\frac{y + d/2}{\ell}\right)^2} \approx \ell\left(1 + \frac{1}{2}\left(\frac{y + d/2}{\ell}\right)^2\right)$$

$$d_2 = \ell\sqrt{1 + \left(\frac{y - d/2}{\ell}\right)^2} \approx \ell\left(1 + \frac{1}{2}\left(\frac{y - d/2}{\ell}\right)^2\right)$$

using $(1+x)^\alpha = 1 + \alpha x$

$$d_1 - d_2 = \ell\Big[1 + \frac{1}{2\ell^2}\left(y^2 + yd + \frac{d^2}{4}\right)\Big)$$
$$- \left(1 + \frac{1}{2\ell^2}\left(y^2 - yd + \frac{d^2}{4}\right)\right)\Big)$$

$$= \ell \cdot \frac{1}{2\ell^2} \cdot 2yd = \frac{yd}{\ell}$$

The minima are located at

$$k(d_1 - d_2) = \frac{kyd}{\ell} = n\pi$$

$$k = \frac{2\pi}{\lambda}$$

$$\boxed{y = \frac{n\lambda}{2} \cdot \frac{\ell}{d}}$$

or more like what you looked up on the internet.

$$\frac{n\lambda}{2} = d \cdot \frac{y}{\ell} = d\theta$$

A5 Mass baseball = 145 g   (5 1/8 oz)

De Broglie Wavelength Baseball

$$\lambda = \frac{h}{p} = \frac{6.63\times10^{-34}\ \text{J s}}{(0.145\ \text{kg})(44\ \text{m/s})} =$$

100 mph = 44 m/s   (WA)

$$\lambda = 1.0\times10^{-34}\ \text{m}$$

From previous problem with $n=1$,

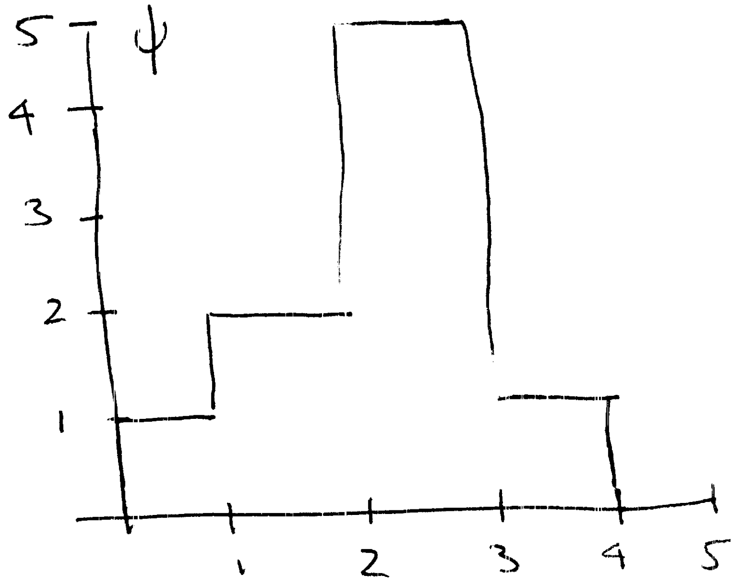
$d = 5\text{ft} = 1.52\ \text{m}$

$$y = \frac{n\lambda}{2}\cdot\frac{\ell}{d} = \frac{(1)\lambda}{2}\cdot\frac{20\text{ft}}{5\text{ft}}$$

$$= 2\lambda = 2\times10^{-34}\ \text{m}$$

sort of hard to see.

A6



<u>Normalize</u>

$\psi' = A\psi$

$1 = \int \psi'^* \psi' \, dx = A^2 \int \psi^2 dx$

$= A^2 \cdot (1 + 4 + 25 + 1) = 31 A^2$

$A = \frac{1}{\sqrt{31}}$

$P(x \in [2,3]) = \int_2^3 \psi'^* \psi' \, dx$

$= \frac{1}{31} \cdot 25$

$= \frac{25}{31}$