PHYS 4073 - Quantum Mechanics- Homework Set 2

Reading Assignment: Chapter 2, TISE, free particles, infinite square well, and delta function potentials

Due at the beginning of class Friday September 10th

Griffiths' Problems

2.4

2.5

2.7

2.8

2.21

2.37 - I'm totally annoyed to be assigning this question but there is not another trig function that does the right stuff.

Additional Problems

Problem A1 - Nano-Wells Wikipedia defines a quantum dot as a structure from 5nm to 50nm. What is the lowest energy state for a 5nm and a 50nm quantum dot treating each as a one-dimensional infinite square well? What is the wavelength of the photon emitted by a transition from the first excited state to the ground state of each well? In what part of the spectrum does the photon fall?

Problem A2 - Stationary States Consider the wave function $\psi(x,0) = A\exp(-ax^2)$. For what potential is this function a stationary state and with what energy?

25 pts each

2.4

$$\phi_n = \sqrt{\frac{2}{L}} \sin k_n x \qquad 0 \le x \le L$$

$$k_n = \frac{n\pi}{L}$$

$$\langle x \rangle = \frac{2}{L}\int_0^L x \sin^2 k_n x \, dx$$

$$= \frac{L}{2}$$

see maple sheet

$$\langle x^2 \rangle = \frac{2}{L}\int_0^L x^2 \sin^2 k_n x \, dx$$

$$= \frac{L^2}{6}\left(2 - \frac{3}{n^2\pi^2}\right)$$

$$\sigma_x = \sqrt{\frac{L^2}{3} - \frac{L^2}{2n^2\pi^2} - \frac{L^2}{4}} = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$= L\sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}}$$

$$\langle p \rangle = \int_0^L \psi^* \frac{\hbar}{i} \frac{\partial}{\partial x} \psi \, dx$$

$$= \frac{k_n \hbar}{i} A^2 \int_0^L \sin k_n x \cos k_n x \, dx = 0$$

$$\langle p^2 \rangle = \int_0^L \psi^* \left(\frac{\hbar}{i}\right)^2 \frac{\partial^2}{\partial x^2} \psi \, dx$$

$$= +\hbar^2 k_n^2 \int A^2 \sin^2 k_n x \, dx$$

$$= \hbar^2 k_n^2 = \frac{\hbar^2 n^2 \pi^2}{L^2}$$

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \frac{\hbar n \pi}{L}$$

<u>Uncertainty</u>

$$\sigma_x \sigma_p = \hbar n \pi \sqrt{\frac{1}{12} - \frac{1}{2n^2\pi^2}}$$

$$= \hbar \sqrt{\frac{n^2\pi^2}{12} - \frac{1}{2}}$$ minimum at $n=1$

At $n=1$ $\sigma_x \sigma_p = \sqrt{\frac{\pi^2}{12} - \frac{1}{2}} = 1.136 \frac{\hbar}{2} > \frac{\hbar}{2}$

$> A := \left(\frac{2}{L}\right)^{\left(\frac{1}{2}\right)};$

$$A := \sqrt{2}\,\sqrt{\frac{1}{L}} \tag{1}$$

$> kn := \frac{n \cdot Pi}{L};$

$$kn := \frac{n\,\pi}{L} \tag{2}$$

$> assume(n :: integer);$

$> int\left(A^2 \cdot sin(kn \cdot x)^2, x = 0 ..L\right);$

$$1 \tag{3}$$

$> int\left(x^2 \cdot A^2 \cdot sin(kn \cdot x)^2, x = 0 ..L\right);$

$$\frac{1}{6}\,\frac{L^2\left(-3 + 2\,n{\sim}^2\,\pi^2\right)}{n{\sim}^2\,\pi^2} \tag{4}$$

$> \left(\frac{h}{I}\right) \cdot int\left(A^2 \cdot sin(kn \cdot x) \cdot diff(sin(kn \cdot x),\ x), x = 0 ..L\right);$

$$0 \tag{5}$$

$> diff(sin(kn \cdot x),\ x);$

$$\frac{\cos\left(\frac{n{\sim}\,\pi\,x}{L}\right) n{\sim}\,\pi}{L} \tag{6}$$

$> \left(\frac{h}{I}\right)^2 \cdot int\left(A^2 \cdot sin(kn \cdot x) \cdot diff(diff(sin(kn \cdot x),\ x), x), x = 0 ..L\right);$

$$\frac{h^2\,n{\sim}^2\,\pi^2}{L^2} \tag{7}$$

$> int\left(x \cdot A^2 \cdot sin(kn \cdot x)^2, x = 0 ..L\right);$

$$\frac{1}{2}\,L \tag{8}$$

$>$

2.5

$$\Psi(x,0) = A(\psi_1 + \psi_2) = A(\phi_1 + \phi_2)$$

(a)

$$1 = |A|^2 \int_{-\infty}^{\infty} (\psi_1^* + \psi_2^*) \cdot (\psi_1 + \psi_2)\, dx$$

$$= |A|^2 \Big[\underbrace{\int \psi_1^* \psi_1\, dx}_{1} + \underbrace{\int \psi_1^* \psi_2\, dx}_{0} + \underbrace{\int \psi_2^* \psi_1\, dx}_{0} + \underbrace{\int \psi_2^* \psi_2\, dx}_{1} \Big]$$

$$= 2|A|^2$$

$$\boxed{A = \frac{1}{\sqrt{2}}} \qquad \Psi(x,0) = \frac{1}{\sqrt{2}}\psi_1 + \frac{1}{\sqrt{2}}\psi_2$$

just like they were vectors.

(b)

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left(\phi_1 e^{-i\omega_1 t} + \phi_2 e^{-i\omega_2 t}\right)$$

$$\omega_i = \frac{E_i}{\hbar} = \frac{k_i^2 \hbar^2}{2m\hbar} = \frac{k_i^2 \hbar}{2m}$$

$$\phi_i = \sqrt{\frac{2}{L}} \sin k_i x$$

$$k_n = \frac{n\pi}{L}$$

$$|\Psi|^2 = \Psi^* \Psi =$$

$$\frac{1}{2}\left(\phi_1 e^{i\omega_1 t} + \phi_2 e^{i\omega_2 t}\right)\left(\phi_1 e^{-i\omega_1 t} + \phi_2 e^{-i\omega_2 t}\right)$$

$$= \frac{1}{2}\left(\phi_1^2 + \phi_2^2 + \phi_1 \phi_2 \left(e^{i(\omega_1 - \omega_2)t} + e^{-i(\omega_1 - \omega_2)t}\right)\right)$$

<u>Eulers</u> $\quad e^{ix} + e^{-ix} = 2\cos x$

$$|\psi|^2 = \frac{1}{2}\left(\phi_1^2 + \phi_2^2 + 2\cos((\omega_1-\omega_2)t)\,\phi_1\phi_2\right)$$

Let $\omega_1 \equiv \omega = \dfrac{\hbar k_1^2}{2m} = \dfrac{\hbar\pi^2}{2mL^2}$

$$\omega_n = n^2\omega$$

$$\omega_1 - \omega_2 = -3\omega$$

$$\boxed{|\psi|^2 = \frac{1}{2}\left(\frac{2}{L}\right)\left[\sin^2\frac{\pi x}{L} + \sin^2\frac{2\pi x}{L} + 2\cos 3\omega t\,\sin\frac{\pi x}{L}\sin\frac{2\pi x}{L}\right]}$$

(c)

$$\langle x\rangle = \int_0^L x|\psi|^2 dx$$

$$= \frac{1}{2}\left[\langle x\rangle_1 + \langle x_2\rangle + \frac{4}{L}\int_0^L x\cos 3\omega t\,\sin\frac{\pi x}{L}\sin\frac{2\pi x}{L}\,dx\right.$$

Back to Maple

$$\int_0^L x \sin\frac{\pi x}{L} \sin\frac{2\pi x}{L}\, dx = -\frac{8}{9}\frac{L^2}{\pi^2}$$

I have defined $\langle x \rangle_1 = \int_0^L x \phi_1^* \phi_1 \, dx = \frac{L}{2}$

from previous problem.

$$\langle x \rangle = \frac{1}{2}\left[\frac{L}{2} + \frac{L}{2} + \frac{4}{L} \cos 3\omega t \left(-\frac{8}{9}\frac{L^2}{\pi^2} \right) \right]$$

$$= \frac{L}{2} - \frac{16L}{9\pi^2} \cos 3\omega t$$

$$= \frac{L}{2}\left(1 - \frac{32}{9\pi^2} \cos 3\omega t \right)$$

Amplitude

$$\frac{32}{9\pi^2}\left(\frac{L}{2}\right) = 0.36\left(\frac{L}{2}\right)$$

Frequency

$$3\omega = \frac{3\hbar\pi^2}{2mL^2}$$

(d)

$$\langle p \rangle = m \frac{d\langle x \rangle}{dt} = \frac{mL}{2}\left(\frac{32}{9\pi^2} \cdot 3\omega \cdot \sin 3\omega t\right)$$

$$= \frac{16mL}{3\pi^2}\,\omega \sin 3\omega t$$

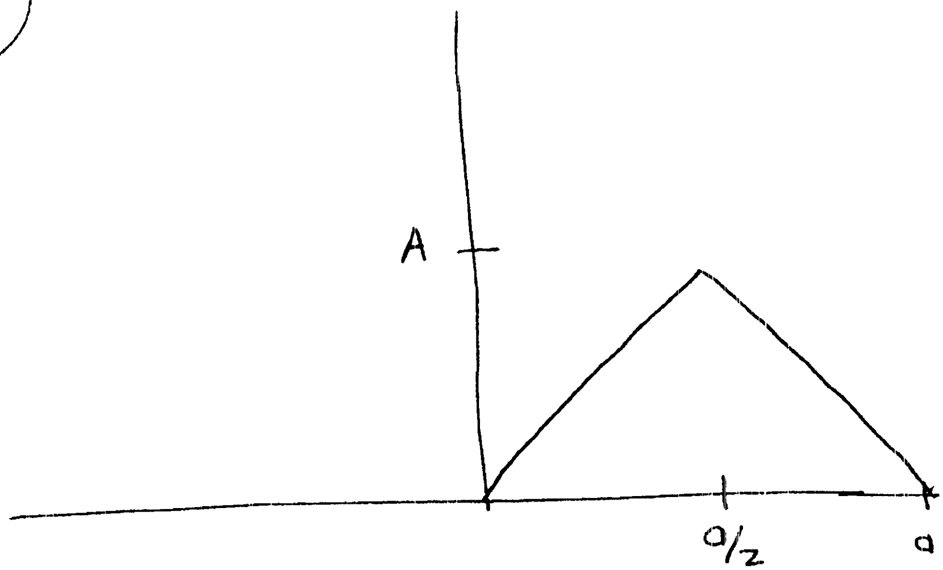
$$= \frac{16mL}{3\pi^2}\left(\frac{\hbar\pi^2}{2mL^2}\right) \times \sin 3\omega t = \frac{8\hbar}{3L}\sin 3\omega t$$

(e) You could observe E_1 or E_2 with equal ~~likelihood~~ probability.

$$\langle H \rangle = \frac{1}{2}E_1 + \frac{1}{2}E_2 = \frac{1}{2}\left(\frac{\hbar^2\pi^2}{2mL^2} + \frac{4\hbar^2\pi^2}{2mL^2}\right)$$

$$= \frac{5\hbar^2\pi^2}{4mL^2}$$

2.7



$$1 = \int \psi^* \psi \, dx = \int_0^{a/2} A^2 x^2 dx + \int_{a/2}^{a} A^2 (a-x)^2 dx$$

$$= \frac{A^2 a^3}{12}$$

$$A = \sqrt{\frac{12}{a^3}}$$

(b) $\psi(x,t) = \sum_i \phi_i e^{-i\omega_i t}$

$$\phi_n = \sqrt{\frac{2}{a}} \sin k_n x$$

$$k_n = \frac{n\pi}{a}$$

$$\omega_n = \frac{E_n}{\hbar} = \frac{\hbar k_n^2}{2m} = \frac{\hbar^2 n^2 \pi^2}{2m}$$

Using orthogonality

$$c_n = \int \phi_n^* \psi(x,0)\, dx$$

$$= \frac{4\sqrt{6}}{n\pi^2} \sin\left(\frac{n\pi}{2}\right)$$

$$c_n = \begin{cases} 0 & n \text{ even} \\ \frac{4\sqrt{6}}{n^2\pi^2}(-1)^{(n-1)/2} & n \text{ odd} \end{cases}$$

$$\psi(x,t) = \sum_n c_n \sqrt{\frac{2}{d}} \sin k_n x \, e^{-i\omega_n t}$$

(c) $P_1 = c_1^* c_1$

$$= \left(\frac{4\sqrt{6}}{\pi^2}\right)^2 = \frac{16 \cdot 6}{\pi^4} = 0.9855$$

so the particle is quite likely to be found in the ground state.

(d)

$$\langle H \rangle = \sum c_n^* c_n H_n$$

$$= \sum_{n=1,3,5} \frac{16 \cdot 6}{n^4 \pi^4} \cdot \frac{\hbar^2}{2m}\left(\frac{n\pi}{a}\right)^2$$

$$= \sum_{n=1,3,5} \frac{48\hbar^2}{ma^2\pi^2} \frac{1}{n^2}$$

$$\sum_{n\text{-odd}} \frac{1}{n^2} = \frac{\pi^2}{8}$$ Schaums

$$\langle H \rangle = \frac{6\hbar^2}{ma^2}$$

2.8

$$\psi(x,0) = \begin{cases} A & x < a/2 \\ 0 & x > a/2 \end{cases}$$

$$1 = \int_0^{a/2} \psi^* \psi \, dx = A^2 \int dx = A^2 \frac{a}{2}$$

$$A = \sqrt{\frac{2}{a}}$$

(b) For the infinite square well

$$E_1 = \frac{\hbar^2 k_1^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2$$

$$= \frac{\hbar^2 \pi^2}{2ma^2} \qquad k_n = \frac{n\pi}{a}$$

So a measurement of the given energy places the particle in th ϕ_1 state

$$\phi_n = \sqrt{\frac{2}{a}} \sin k_n x$$

The wave function can be expanded in terms of the stationary solutions.

$$\psi(x,0) = \sum c_n \phi_n$$

and the probability of the nth solution becomes

$$P_n = c_n^* c_n$$

So we need c_1

$$c_1 = \int_0^a \phi_1^* \psi(x,0)\, dx$$

$$= \left(\sqrt{\frac{2}{a}}\right)\left(\sqrt{\frac{2}{a}}\right) \int_0^{a/2} \sin k_1 x \, dx$$

$$= \frac{2}{a k_1}\left(-\cos k_1 x\right)\Big|_0^{a/2} \qquad k_1 = \frac{\pi}{a}$$

$$= \frac{2}{k_1 a}\left(1 - \cos\frac{k_1 a}{2}\right) = \frac{2}{\pi}\left(1 - \cos\left(\frac{\pi}{2}\right)\right)$$

$$= \frac{2}{\pi}$$

So the probability is

$$P_1 = C_1^* C_1 = \left(\frac{2}{\pi}\right)^2 = 0.41$$

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$$\psi(x) = A e^{-a|x|}$$

Normalize

$$1 = A^2 \int_{-\infty}^{\infty} e^{-2a|x|} \, dx$$

$$= 2A^2 \int_0^{\infty} e^{-2ax} \, dx$$

$$= \frac{2A^2}{-2a} e^{-2ax} \Big|_0^{\infty} = \frac{A^2}{a}$$

$\cancel{A = \frac{1}{\sqrt{a}}}$ $\quad A = \sqrt{a}$

$$\psi(x,0) = \sqrt{a} \, e^{-a|x|}$$

(b) $$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x,0)\, e^{-ikx}\, dx$$

$$= \frac{\sqrt{a}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|}\, e^{-ikx}\, dx$$

$$= \sqrt{\frac{a}{2\pi}} \left[\int_{-\infty}^{0} e^{+ax-ikx}\, dx + \int_{0}^{\infty} e^{-ax-ikx}\, dx \right]$$

$$\int_0^{\infty} e^{-bx}\, dx = \frac{1}{-b} e^{-bx} \Big|_0^{\infty} = \frac{1}{b}$$

$$\int_{-\infty}^{0} e^{bx}\, dx = \frac{1}{b} e^{bx} \Big|_{-\infty}^{0} = \frac{1}{b}$$

$$\phi(k) = \sqrt{\frac{a}{2\pi}} \left[\frac{1}{a-ik} + \frac{1}{a+ik} \right]$$

$$= \sqrt{\frac{a}{2\pi}} \left[\frac{a+ik+a-ik}{a^2+k^2} \right]$$

$$= 2a\sqrt{\frac{a}{2\pi}}\, \frac{1}{a^2+k^2}$$

(c)

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int \phi(k)\, e^{ikx}\, e^{-i\omega t}\, dk$$

Dispersion Relation Free Particle

$$E = \hbar\omega = \frac{\hbar^2 k^2}{2m}$$

$$\omega = \frac{\hbar k^2}{2m}$$

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \cdot 2a \cdot \sqrt{\frac{a}{2\pi}} \int_{-\infty}^{\infty} dk\, \frac{1}{a^2+k^2}\, e^{ikx}\, e^{-\frac{i\hbar k^2 t}{2m}}$$

(d) ~~If a large $e^{-a|x|}$ broad and~~ the ~~position is uncertain, but $1/a^2+k^2$~~

If "a" large, $e^{-a|x|}$ is very narrow and the position is well defined. If "a" large then $1/a^2+k^2$ is broad and the momentum is poorly defined.

The opposite is true if a small.

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$$\psi(x,0) = A \sin^3(\pi x/a)$$

$$= \frac{A}{4}\left(3\sin\frac{\pi x}{a} - \sin\frac{3\pi x}{a}\right)$$

$$= A'\left(3\sqrt{\frac{2}{a}}\sin k_1 x - \sqrt{\frac{2}{a}}\sin 3k_1 x\right)$$

$$\cancel{= A'(\phi_1(x)} \quad = A'(3\phi_1 - \phi_3)$$

where $\phi_i = \sqrt{\frac{2}{a}} \sin k_i x$

$$k_n = \frac{n\pi}{a}$$

In terms of ϕ_i, the "length" of ψ is

$$\sqrt{A'^2(3^2+1^2)} = \sqrt{10}\, A'$$

$$\psi(x,0) = \frac{3}{\sqrt{10}}\phi_1 - \frac{1}{\sqrt{10}}\phi_3$$

Time Evolution

$$\psi(x,t) = \frac{3}{\sqrt{10}}\,\phi_1(x)\,e^{-i\omega_1 t} - \frac{1}{\sqrt{10}}\,\phi_3(x)\,e^{-i\omega_3 t}$$

$$\omega_n = \frac{\hbar k_n^2}{2m} = n^2 \underbrace{\left(\frac{\hbar \pi^2}{2ma^2}\right)}_{\omega_0}$$

$$= n^2 \omega_0$$

$$\psi(x,t) = \frac{3}{\sqrt{10}}\,\phi_1\,e^{-i\omega_0 t} - \frac{1}{\sqrt{10}}\,\phi_3(x)\,e^{-3i\omega_0 t}$$

Expectation Value $\langle x \rangle$

$$\langle x \rangle = \int x\,\psi^* \psi\,dx$$

$$= \int x \left(\frac{3}{\sqrt{10}}\,\phi_1^*\,e^{i\omega_0 t} - \frac{1}{\sqrt{10}}\,\phi_3^*\,e^{3i\omega_0 t}\right) \cdot \left(\frac{3}{\sqrt{10}}\,\phi_1\,e^{-i\omega_0 t} - \frac{1}{\sqrt{10}}\,\phi_3\,e^{-3i\omega_0 t}\right)$$

$$\langle x \rangle = \frac{9}{10}\int x\,\phi_1^*\,\phi_1\,dx + \frac{1}{10}\int \phi_3^*\phi_3\,x\,dx$$

$$- \frac{3}{10}\int \left[x\,\phi_1^*\phi_3\, e^{-2i\omega_0 t} + x\,\phi_3^*\phi_1\, e^{2i\omega_0 t} \right] dx$$

$\int x\,\phi_1^*\,\phi_1\,dx =$ Expectation Value x in ground state

$= \frac{a}{2}$

$$\int x\,\phi_3^*\phi_3\,dx = a/2$$

$$\int x\,\phi_1^*\phi_3\,dx = \int x\,\phi_1\,\phi_3 = \frac{2}{a}\int_0^a x \sin\frac{\pi x}{a}\,\sin\frac{3\pi x}{a}\,dx$$

$= 0$ (Wolfram Alpha)

$$\langle x \rangle = \frac{9}{10}\frac{a}{2} + \frac{1}{10}\frac{a}{2} = \frac{a}{2}$$

(A1) Energy of infinite square well

$$E_i = \frac{\hbar^2 k_i^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{n\pi}{d}\right)^2 = n^2 E_0$$

$$\cancel{\Delta E} \; E_0 \equiv \frac{\hbar^2 \pi^2}{2ma^2} = \frac{(1.05\times10^{-34}\,J\cdot s)^2\,\pi^2}{2(9.11\times10^{-31}\,Kg)(5\times10^{-9}\,m)^2}$$

Lowest Energy 5nm $= 2.39\times10^{-21}\,J$

Lowest Energy 50nm $= 2.39\times10^{-21}\,J/100 = 2.39\times10^{-23}\,J$

Energy of Photon for transition first excited state to ground state

$$\Delta E = E_2 - E_1 = 4E_0 - E_0 = 3E_0$$

$$= 7.17\times10^{-21}\,J \quad (5nm)$$

$$= \cancel{Xfcz}\; \hbar\omega$$

Frequency $\omega = \frac{\Delta E}{\hbar} = \frac{7.17\times10^{-21}\,J}{1.05\times10^{-34}\,J\cdot s} = 6.83\times10^{13}\,Hz$

$$f = \frac{\omega}{2\pi} = 1.08\times10^{13}\,Hz$$

$$\lambda = \frac{c}{f} = \frac{3 \times 10^{8}\ m/s}{1.09 \times 10^{13}\ s^{-1}} = 2.76 \times 10^{-5}\ m$$

at $a = 5nm$

$= 27.6\ \mu m$

infrared

$= 27,600\ nm$

The 50nm well produces longer wavelengths lower energy

$$\lambda_{50nm} = 2.76 \times 10^{-3}\ m = 2.7mm$$

This sits right at the edge of the ~~far~~ infrared / microwave spectrum.

A2

$$\frac{d^2}{dx^2} A e^{-ax^2} = A\left(4a^2x^2e^{-ax^2} - 2ae^{-ax^2}\right)$$

Wolfram Alpha

$$= 4a^2x^2\psi - 2a\psi$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi - V\psi$$

$$= +\frac{2a\hbar^2}{2m}\psi - \frac{4a^2x^2\hbar^2}{2m}\psi$$

$$E = \frac{a\hbar^2}{m} \qquad V(x) = \frac{2a^2\hbar^2}{m}x^2$$

Another simple harmonic oscillator