PHYS 4073 - Quantum Mechanics- Homework Set 3

Reading Assignment: Chapter 2, remaining material except the simple harmonic oscillator

Due at the beginning of class Friday September 24th. Test 1 will be from 6-9pm on Weds. September 29th.

Griffiths' Problems

2.23

2.24

2.29

2.30

2.33 Work $E > V_0$ case

Additional Problems

Problem A1 - Quantum Tuning A stream of electrons is incident on two metal plates. The plates are spaced 1mm apart. An electric potential difference of 3000V is established across the plates. This is the largest potential available before sparking. You naturally will realize that the V in the SE is electric potential energy and recall the relation between potential and potential energy. You wish to use this setup to adjust the flux of electrons with energy E. The electrons are incident from the low potential side and since the screen is narrow it may be treated as a step function. How fast must the electrons be travelling for 50% transmission?

Problem A2 - Double Delta Function Wells Consider the double delta function well $V(x) = -\alpha\delta(x - a/2) - \alpha\delta(x + a/2)$ where α is a positive constant. Write the wave function $\phi(x)$ and report all equations of the coefficients resulting from applying the boundary conditions. You do not need to solve for the coefficients. Do $E<0$ states.

Problem A3 - Finite Step In lecture I worked the finite step ($V = 0$ for $x < 0$ and $V = V_0$ for $x > 0$ for $E > V_0$) and then pulled from slight of hand with complex numbers to solve the $E < V_0$ case. Work the $E < V_0$ case of the finite step potential from scratch.

Problem A4 - Finite Well Deuterium is an isotope of hydrogen with one proton and one neutron. The binding energy of deuterium is 2.24MeV and the nuclear radius is $a/2 = 2.14$fm. Treating the binding of deuterium as a single particle trapped in an infinite square well of width a, what is the minimum energy of the proton in deuterium? How does this answer change if we model the well as a finite well of height 2.24MeV, that is calculate the finite well energy.

25 points each. To be returned Monday.

2.23

(a) = $\int_{-3}^{1} (x^3 - 3x^2 + 2x - 1)\,\delta(x+2)\,dx$

$= (-2)^3 - 3(-2)^2 + (2)(-2) - 1$

$-8 \quad -12 \quad -4 \quad -1 \quad = -25$

(b) $\int_{0}^{\infty} [\cos 3x + 2]\,\delta(x-\pi)\,dx$

$= \cos 3\pi + 2 = 1$

(c) $\int_{-1}^{+1} \exp(|x|+3)\,\delta(x-2)\,dx = 0$

$2 \notin [-1, 1]$

(2.24)

(a) $$\int_{-\infty}^{\infty} \delta(cx) f(x)\, dx = \begin{cases} \frac{1}{c} \int_{-\infty}^{\infty} \delta(u) f\left(\frac{u}{c}\right) du & c>0 \\ \frac{1}{c} \int_{\infty}^{-\infty} \delta(u) f\left(\frac{u}{c}\right) du & c<0 \end{cases}$$

$$u = cx$$

$$\int_{-\infty}^{\infty} \delta(cx) f(x)\, dx = \frac{1}{|c|} \int_{-\infty}^{\infty} \delta(u) f\left(\frac{u}{c}\right) du$$

$$= \frac{f(0)}{|c|}$$

$$= \int_{-\infty}^{\infty} \frac{\delta(x)}{|c|} f(x)\, dx$$

(b) $$\delta(x) = \frac{d\theta}{dx}$$

$$\int_{-\infty}^{\infty} \delta(x) f(x)\, dx = f(0)$$

$$\int_{-\infty}^{\infty} \frac{d\Theta}{dx} f(x)\, dx = \Theta f \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{df}{dx} \Theta \, dx$$

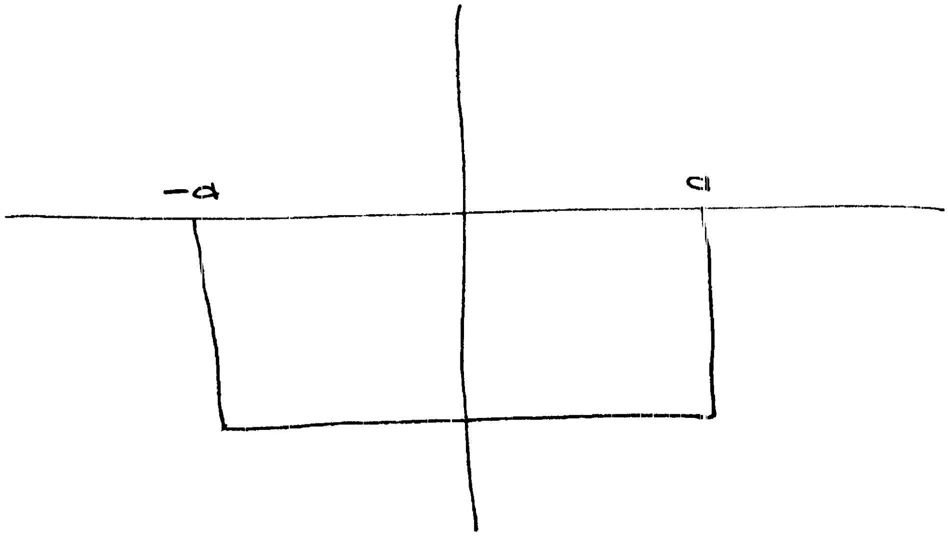
integration by parts

$$= f(\infty) - \int_{0}^{\infty} \frac{df}{dx}\, dx$$

$$= f(\infty) - f \Big|_{0}^{\infty}$$

$$= f(\infty) - f(\infty) + f(0) = f(0)$$ ✓

2.29



Odd solutions

$$\phi(x) = \begin{cases} F e^{-\kappa x} & x > a \\ D \sin lx & -a < x < a \\ -\phi(-x) & x < 0 \end{cases}$$

Eqn 2.151

Continuity

$$F e^{-\kappa a} = D \sin la$$

Slope Continuity

$$-\kappa F e^{-\kappa a} = +Dl\cos(la)$$

Divide two equations

$$-\kappa = l \cot la$$

$$\kappa = \sqrt{\frac{-2mE}{\hbar^2}} \qquad l = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}$$

Solve for E

Let $z \equiv la$

$$-\kappa a = z \cot z$$

$$l^2 = \frac{2mE}{\hbar^2} + \frac{2mV_0}{\hbar^2}$$

$$z^2 = \frac{2mEa^2}{\hbar^2} + \frac{2mV_0 a^2}{\hbar^2}$$

Define $z_0 = \frac{a}{\hbar}\sqrt{2mV_0}$

$$k^2a^2 = -\frac{2mEa^2}{\hbar^2}$$

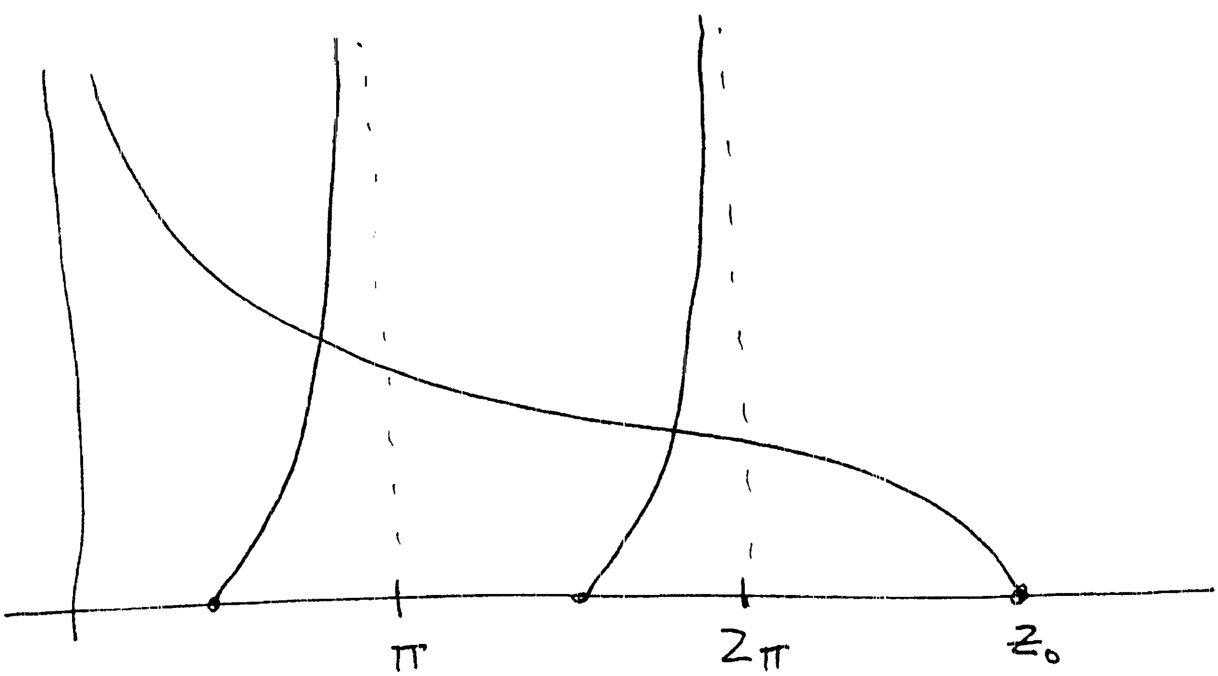
$$z^2 = -k^2a^2 + z_0^2$$

$$ka = \sqrt{z_0^2 - z^2}$$

Our equation for the energy becomes

$$\sqrt{z_0^2 - z^2} = -z\cot z$$

$$-\cot z = \sqrt{\left(\frac{z_0}{z}\right)^2 - 1}$$



Wide, deep well (z_0 large) – Intersections happen at $\pi, 2\pi, 3\pi \ldots$

Shallow, narrow well If $z_0 < \frac{\pi}{2}$ there are no odd solutions

$$z_0 = \frac{a}{\hbar}\sqrt{2mV_0} < \pi/2$$

$$\frac{a^2}{\hbar^2} 2mV_0 < \frac{\pi^2}{4}$$

$$V_0 < \frac{\hbar^2 \pi^2}{8ma^2} \quad \text{no solutions.}$$

(2.30)

$$\phi(x) = \begin{cases} F e^{-kx} & x > a \\ D\cos(\ell x) & -a < x < a \\ -F e^{kx} & x < -a \end{cases}$$

Since ϕ even, probability is symmetric

$$\int_{-\infty}^{\infty} \phi^* \phi \, dx = 2 \int_0^{\infty} \phi^* \phi \, dx = 1$$

$$\int_0^{\infty} \phi^* \phi \, dx = \frac{1}{2} = D^2 \int_0^a \cos^2(\ell x)\, dx + F^2 \int_a^{\infty} e^{-2kx}\, dx$$

$$\frac{1}{2} = \frac{D^2}{4\ell} \cdot \left(2a\ell + \sin(2a\ell)\right) + F^2 \cdot \frac{e^{-2ak}}{2k}$$

From continuity,

$$F e^{-ka} = D\cos(\ell a)$$

$$F^2 e^{-2ka} = D^2 \cos^2(la)$$

$$\frac{1}{2} = \frac{D^2}{4l}\left(2al + \sin(2al)\right) + \frac{D^2\cos^2(la)}{2k}$$

$$\frac{1}{D^2} = \frac{1}{2l}\left(2al + \sin(2al)\right) + \frac{\cos^2(la)}{k}$$

$$= \frac{1}{2kl}\left(2akl + k\sin(2al) + 2l\cos^2 la\right)$$

$$D^2 = \frac{2kl}{2akl + k\sin(2al) + 2l\cos^2 la}$$

<u>Simplify using</u> $k = l\tan(la)$

$$D^2 = \frac{2l^2\tan(la)}{2al^2\tan(la) + 2l\tan(la)\sin(la)\cos(la) + 2l\cos^2 la}$$

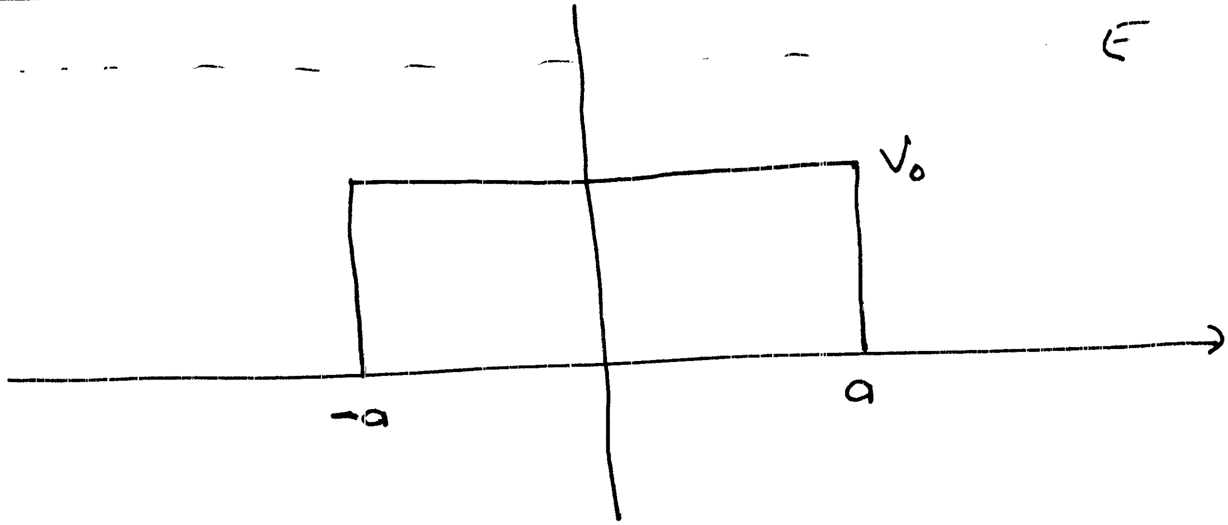
$$\tan(la) = \sin(la)/\cos(la)$$

$$D^2 = \frac{l \sin(la)/\cos(la)}{al \sin(la)/\cos(la) + \sin^2(la) + \cos^2(la)}$$

$$D^2 = \frac{l}{al + \cot(la)}$$

$$F^2 = D^2 e^{2ka} \cos^2(la)$$

(2.33) Finite Barrier



$$k = \sqrt{\frac{2mE}{\hbar^2}} \qquad l = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$$

$$\phi(x) = \begin{cases} A e^{ikx} + B e^{-ikx} & x < -a \\ C e^{ilx} + D e^{-ilx} & -a < x < a \\ F e^{ikx} & x > a \end{cases}$$

Boundary Conditions

Continuous $x = -a$

(1) $$Ae^{-ika} + Be^{ika} = Ce^{-ila} + De^{ila}$$

Continuous $x = +a$

(2) $$Ce^{ila} + De^{-ila} = Fe^{ika}$$

Slope Continuous $x = -a$

(3) $$ikAe^{-ika} - ikBe^{ika} = ilCe^{-ila} - ilDe^{ila}$$

Slope Continuous $x = +a$

(4) $$ilCe^{ila} - ilDe^{-ila} = ikFe^{ika}$$

$i\ell (2) + (4)$

$$2i\ell C e^{i\ell a} = i(k+\ell) F e^{ika}$$

$$(5) \quad C = \frac{1}{2}\left(1+\frac{k}{\ell}\right) F e^{ika} e^{-i\ell a}$$

$-i\ell (2) + (4)$

$$-2i\ell D e^{-i\ell a} = i(k-\ell) F e^{ika}$$

$$(6) \quad D = \frac{1}{2}\left(1-\frac{k}{\ell}\right) F e^{ika} e^{i\ell a}$$

$ik (1) + (3)$

$$2ikA e^{-ika} = i(k+\ell) C e^{-i\ell a} + i(k-\ell) D e^{i\ell a}$$

$$(7) \quad A = \frac{1}{2} e^{ika}\left[\left(1+\frac{\ell}{k}\right) C e^{-i\ell a} + \left(1-\frac{\ell}{k}\right) D e^{i\ell a}\right]$$

Substitute 5, 6 $\rightarrow$ 7

$$A = \frac{1}{2} e^{ika} \Big[\left(1 + \frac{l}{k}\right) \frac{1}{2} \left(1 + \frac{k}{l}\right) F e^{ika} e^{-ila} e^{-ila}$$

$$+ \left(1 - \frac{l}{k}\right) \frac{1}{2} \left(1 - \frac{k}{l}\right) F e^{ika} e^{ila} e^{ila} \Big]$$

$$A = \frac{1}{4} F e^{2ika} \Big[\left(1 + \frac{l}{k}\right)\left(1 + \frac{k}{l}\right) e^{-2ila} +$$

$$\left(1 - \frac{l}{k}\right)\left(1 - \frac{k}{l}\right) e^{2ila} \Big]$$

$$= \frac{1}{4} F e^{2ika} \Big[\left(1 + 1 + \frac{l}{k} + \frac{k}{l}\right) e^{-2ila}$$

$$+ \left(1 + 1 - \frac{l}{k} - \frac{k}{l}\right) e^{2ila} \Big]$$

$$= \frac{1}{4} F e^{2ika} \Big[2\left(e^{2ila} + e^{-2ila}\right)$$

$$+ \left(\frac{l}{k} + \frac{k}{l}\right)\left(e^{-2ila} - e^{2ila}\right)$$

$$e^{2ila} + e^{-2ila} = 2\cos(2la)$$

$$e^{-2ila} - e^{2ila} = -2i\sin(2la)$$

$$A = \frac{1}{4} F e^{2ika}\left[4\cos(2la) - 2i\underbrace{\left(\frac{l}{k} + \frac{k}{l}\right)}_{\frac{l^2+k^2}{kl}}\sin(2la)\right]$$

$$A = F e^{2ika}\left[\cos(2la) - i\left(\frac{l^2+k^2}{2kl}\right)\sin(2la)\right]$$

$$\frac{1}{T} = \frac{A^*A}{F^*F} = \underbrace{\cos^2(2la)}_{1-\sin^2(2la)} + \left(\frac{l^2+k^2}{2kl}\right)^2\sin^2(2la)$$

$$\frac{1}{T} = 1 + \underbrace{\left(\left[\frac{l^2+k^2}{2kl}\right]^2 - 1\right)}_{\frac{l^4+2l^2k^2+k^4-4k^2l^2}{(2kl)^2}}\sin^2(2la)$$

$$\frac{1}{T} = 1 + \left(\frac{(l^2 - k^2)^2}{(2kl)^2}\right) \sin^2(2la)$$