PHYS 4073 - Quantum Mechanics- Homework Set 4

Reading Assignment: Appendix A

Due at the beginning of class Friday October 8th. This assignment will be really painful without math software.

Griffiths' Problems

A.1

A.4

A.6

A.8

A.19

A.25 parts a-c

A.26 parts a-c ~~Grade~~

~~25 points each~~

25 points each.

(A1)

(a) The vectors inherit all the associative, distributive stuff from the normal 3 space vectors so all we have to check is:

(1) Vector addition leaves the vector in the space

$$\vec{V}_1 + \vec{V}_2 = (a, b, 0) + (c, d, 0)$$
$$= (a+b, c+d, 0) \in \mathbb{V}_{z=0} \quad ✓$$

(2) Scalar multiplication must leave the vector in the space

$$c(a, b, 0) - (ac, ab, 0) \in \mathbb{V}_{z=0}$$

(3) The null vector must be in the space

$$\vec{0} = (0, 0, 0) \in \mathbb{V}_{z=0}$$

(4) Inverse vector must be in the space

$$-\vec{v} = (-a, -b, 0) \in \mathbb{V}_{z=0} \quad ✓$$

$\mathbb{V}_{z=0}$ is a vector space spanned by $(1,0,0)$ and $(0,1,0)$ $\Longrightarrow$ $\text{Dim} = 2$

(b)

Check vector addition,

$$(a,b,1) + (c,d,1) = (a+c, b+d, 2) \notin \mathbb{V}_{z=1}$$

$\Rightarrow$ Not a vector space

(c) Let $\mathbb{V}_=$ be possible vector space with all components equal. Once again associative and distributive inherited from 3 space

Check Vector Addition

$$\vec{v}_1 + \vec{v}_2 = (a, a, a) + (b, b, b) = (a+b, a+b, a+b) \in \mathbb{V}_= \checkmark$$

Check Scalar Multiplication

$$c\vec{v}_1 = c(a, a, a)$$

$$= (ca, ca, ca) \in \mathbb{V}_= \checkmark$$

Check Null Vector

$$(0,0,0) \in \mathbb{V}_=$$

Check Inverse

$$-\vec{v} = (-a, -a, -a) \in \mathbb{V}_=$$

$\Rightarrow \mathbb{V}_=$ vector space spanned by $(1,1,1) \Rightarrow$ dim 1.

(A4) The problem becomes much easier if we re-order the vectors.

$$|a_1\rangle = 28\hat{y}$$

$$|a_2\rangle = i\hat{x} + 3\hat{y} + \hat{z}$$

$$|a_3\rangle = (1+i)\hat{x} + \hat{y} + i\hat{z}$$

$$|e_1\rangle = \frac{|a_1\rangle}{\sqrt{\langle a_1|a_1\rangle}} = \hat{y}$$

$$|b_2\rangle = |a_2\rangle - \langle e_1|a_2\rangle |e_1\rangle$$

$$= i\hat{x} + \hat{z}$$

$$\langle b_2|b_2\rangle = (-i\hat{x} + \hat{z}) \cdot (i\hat{x} + \hat{z})$$

$$= 1 + 1 = 2$$

$$|e_2\rangle = \frac{|b_2\rangle}{\sqrt{\langle b_2|b_2\rangle}} = \frac{i}{\sqrt{2}}\hat{x} + \frac{1}{\sqrt{2}}\hat{z}$$

$$|b_3\rangle = |a_3\rangle - \langle e_1|a_3\rangle|e_1\rangle$$
$$- \langle e_2|a_3\rangle|e_2\rangle$$

$$= (1+i)\hat{x} + \hat{y} + i\hat{z}$$
$$- \left[\hat{y}\cdot\left((1+i)\hat{x} + \hat{y} + i\hat{z}\right)\right]\hat{y}$$
$$- \left[\left(-\frac{i}{\sqrt{2}}\hat{x} + \frac{1}{\sqrt{2}}\hat{z}\right)\cdot\left((1+i)\hat{x} + \hat{y} + i\hat{z}\right)\right]\left(\frac{i}{\sqrt{2}}\hat{x} + \frac{1}{\sqrt{2}}\hat{z}\right)$$

$$= (1+i)\hat{x} + \hat{y} + i\hat{z} - \hat{y}$$
$$- \underbrace{\left[\frac{1}{\sqrt{2}}(1-i) + \frac{i}{\sqrt{2}}\right]\frac{1}{\sqrt{2}}}_{\frac{1}{2}}(i\hat{x} + \hat{z})$$

$$= (1+i)\hat{x} + i\hat{z} - \frac{1}{2}i\hat{x} - \frac{1}{2}\hat{x}$$

$$|b_2\rangle = \left(1 + \frac{i}{2}\right)\hat{x} + \left(i - \frac{1}{2}\right)\hat{z}$$

$$\langle b_2|b_2\rangle = 1 + \frac{1}{4} + 1 + 1/4 = 5/2$$

$$|e_3\rangle = \frac{|b_3\rangle}{\sqrt{\langle b_3|b_3\rangle}} = \sqrt{\frac{2}{5}}\left[\left(1+\frac{i}{2}\right)\hat{x} + \left(i-\frac{1}{2}\right)\hat{z}\right]$$

<u>Check</u>

$$\langle e_2|e_3\rangle = \frac{1}{\sqrt{2}}\left(-i\hat{x} + \hat{z}\right)\sqrt{\frac{2}{5}}\left[\left(1+\frac{i}{2}\right)\hat{x} + \left(i-\frac{1}{2}\right)\hat{z}\right]$$

$$= \sqrt{\frac{1}{5}}\left[-i + \frac{1}{2} + i - \frac{1}{2}\right] = 0 \quad \checkmark$$

$$\langle e_1|e_2\rangle = 0$$

$$\langle e_1|e_3\rangle = 0$$

Griffiths' Solution for order given in problem.

Problem A.4

(i)

$$\langle e_1|e_1\rangle = |1+i|^2 + 1 + |i|^2 = (1+i)(1-i) + 1 + (i)(-i) = 1+1+1+1 = 4. \quad \|e_1\| = 2.$$

$$\boxed{|e_1'\rangle = \frac{1}{2}(1+i)\,\hat{\imath} + \frac{1}{2}\,\hat{\jmath} + \frac{i}{2}\,\hat{k}.}$$

(ii)

$$\langle e_1'|e_2\rangle = \frac{1}{2}(1-i)(i) + \frac{1}{2}(3) + \left(\frac{-i}{2}\right)1 = \frac{1}{2}(i+1+3-i) = 2$$

$$|e_2''\rangle = |e_2\rangle - \langle e_1'|e_2\rangle\,|e_1'\rangle = (i-1-i)\hat{\imath} + (3-1)\hat{\jmath} + (1-i)\hat{k} = (-1)\hat{\imath} + (2)\hat{\jmath} + (1-i)\hat{k}$$

$$\langle e_2''|e_2''\rangle = 1 + 4 + 2 = 7. \quad \boxed{|e_2'\rangle = \frac{1}{\sqrt{7}}\left[-\hat{\imath} + 2\hat{\jmath} + (1-i)\hat{k}\right].}$$

(iii)

$$\langle e_1'|e_3\rangle = \frac{1}{2}28 = 14; \quad \langle e_2'|e_3\rangle = \frac{2}{\sqrt{7}}28 = 8\sqrt{7}.$$

$$\begin{aligned} |e_3''\rangle &= |e_3\rangle - \langle e_1'|e_3\rangle|e_1'\rangle - \langle e_2'|e_3\rangle|e_2'\rangle = |e_3\rangle - 7|e_1\rangle - 8|e_2''\rangle \\ &= (0-7-7i+8)\hat{\imath} + (28-7-16)\hat{\jmath} + (0-7i-8+8i)\hat{k} = (1-7i)\hat{\imath} + 5\hat{\jmath} + (-8+i)\hat{k}. \end{aligned}$$

$$\|e_3''\|^2 = 1 + 49 + 25 + 64 + 1 = 140. \quad \boxed{|e_3'\rangle = \frac{1}{2\sqrt{35}}\left[(1-7i)\hat{\imath} + 5\hat{\jmath} + (-8+i)\hat{k}\right]}$$

Problem A.5

From Eq. A.21: $\langle\gamma|\gamma\rangle = \langle\gamma|\left(|\beta\rangle - \frac{\langle\alpha|\beta\rangle}{\langle\alpha|\alpha\rangle}|\alpha\rangle\right) = \langle\gamma|\beta\rangle - \frac{\langle\alpha|\beta\rangle}{\langle\alpha|\alpha\rangle}\langle\gamma|\alpha\rangle.$ From Eq. A.19,

$$\langle\gamma|\beta\rangle^* = \langle\beta|\gamma\rangle = \langle\beta|\left(|\beta\rangle - \frac{\langle\alpha|\beta\rangle}{\langle\alpha|\alpha\rangle}|\alpha\rangle\right) = \langle\beta|\beta\rangle - \frac{\langle\alpha|\beta\rangle}{\langle\alpha|\alpha\rangle}\langle\beta|\alpha\rangle = \langle\beta|\beta\rangle - \frac{|\langle\alpha|\beta\rangle|^2}{\langle\alpha|\alpha\rangle},$$ which is *real*.

$$\langle\gamma|\alpha\rangle^* = \langle\alpha|\gamma\rangle = \langle\alpha|\left(|\beta\rangle - \frac{\langle\alpha|\beta\rangle}{\langle\alpha|\alpha\rangle}|\alpha\rangle\right) = \langle\alpha|\beta\rangle - \frac{\langle\alpha|\beta\rangle}{\langle\alpha|\alpha\rangle}\langle\alpha|\alpha\rangle = 0. \quad \langle\gamma|\alpha\rangle = 0.$$ So (Eq. A.20):

$$\langle\gamma|\gamma\rangle = \langle\beta|\beta\rangle - \frac{|\langle\alpha|\beta\rangle|^2}{\langle\alpha|\alpha\rangle} \geq 0,$$ and hence $|\langle\alpha|\beta\rangle|^2 \leq \langle\alpha|\alpha\rangle\langle\beta|\beta\rangle$. QED

A6 $|\alpha\rangle = (1+i)\hat{x} + \hat{y} + i\hat{z}$

$|\beta\rangle = (4-i)\hat{x} + (2-2i)\hat{z}$

$\langle\alpha|\alpha\rangle = 2+1+1 = 4$

$\langle\beta|\beta\rangle = 16+1+4+4 = 25$

$\langle\alpha|\beta\rangle = \left[(1-i)\hat{x} + \hat{y} - i\hat{z}\right]\left[(4-i)\hat{x} + (2-2i)\hat{z}\right]$

$= 4 - i - 4i - 1 - 2i - 2 = 1 - 7i$

$\langle\alpha|\beta\rangle\langle\beta|\alpha\rangle = 1 + 49 = 50$

$$\cos\theta = \sqrt{\frac{\langle\alpha|\beta\rangle\langle\beta|\alpha\rangle}{\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle}} = \sqrt{\frac{50}{4 \cdot 25}}$$

$$= \sqrt{\frac{1}{2}}$$

$\theta = \frac{\pi}{4}$

(A8) I did this problem almost entirely with Maple.

(a) $$A+B = \begin{pmatrix} +1 & 1 & 0 \\ 2 & 1 & 3 \\ 3i & 3-2i & 4 \end{pmatrix}$$

(b) AB

$$\begin{pmatrix} -3 & 1+3i & 3i \\ 4+3i & 9 & 6-2i \\ 6i & 6-2i & 6 \end{pmatrix}$$

(c) $[A,B] = AB - BA$

$$\begin{pmatrix} -3 & 1+3i & 3i \\ 2+3i & 9 & 3-2i \\ -6+3i & 6+i & -6 \end{pmatrix}$$

(d)

$$\tilde{A} = \begin{pmatrix} -1 & 2 & 2i \\ 1 & 0 & -2i \\ i & 3 & 2 \end{pmatrix}$$

(e) $A^* = \tilde{A}^+$ (Maple didn't know how to do A^*)

Note: A^+ is Hermitian Transpose (A)

$$A^* = \begin{pmatrix} -1 & 1 & -i \\ 2 & 0 & 3 \\ -2i & 2i & 2 \end{pmatrix}$$

(f)

$$A^+ = \begin{pmatrix} -1 & 2 & -2i \\ 1 & 0 & 2i \\ -i & 3 & 2 \end{pmatrix}$$

(g) $\det(B) = 3$

(h)

$$B^{-1} = \begin{pmatrix} \frac{2}{3} & -i & \frac{1}{3}i \\ 0 & 1 & 0 \\ -\frac{1}{3}i & -2 & \frac{2}{3} \end{pmatrix}$$

$B^{-1}B = 1$ checked with Maple

$\det(A) = 0$ so no inverse.

$> A := \begin{bmatrix} -1 & 1 & I \\ 2 & 0 & 3 \\ 2\cdot I & -2\cdot I & 2 \end{bmatrix};$

$$A := \begin{bmatrix} -1 & 1 & \mathrm{I} \\ 2 & 0 & 3 \\ 2\,\mathrm{I} & -2\,\mathrm{I} & 2 \end{bmatrix} \tag{1}$$

$> B := \begin{bmatrix} 2 & 0 & -I \\ 0 & 1 & 0 \\ I & 3 & 2 \end{bmatrix};$

$$B := \begin{bmatrix} 2 & 0 & -\mathrm{I} \\ 0 & 1 & 0 \\ \mathrm{I} & 3 & 2 \end{bmatrix} \tag{2}$$

$> A + B;$

(a)

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 3 \\ 3\,\mathrm{I} & 3-2\,\mathrm{I} & 4 \end{bmatrix} \tag{3}$$

> *with*(*LinearAlgebra*);

[*&x, Add, Adjoint, BackwardSubstitute, BandMatrix, Basis, BezoutMatrix, BidiagonalForm, BilinearForm, CharacteristicMatrix, CharacteristicPolynomial, Column, ColumnDimension, ColumnOperation, ColumnSpace, CompanionMatrix, ConditionNumber, ConstantMatrix, ConstantVector, Copy, CreatePermutation, CrossProduct, DeleteColumn, DeleteRow, Determinant, Diagonal, DiagonalMatrix, Dimension, Dimensions, DotProduct, EigenConditionNumbers, Eigenvalues, Eigenvectors, Equal, ForwardSubstitute, FrobeniusForm, GaussianElimination, GenerateEquations, GenerateMatrix, GetResultDataType, GetResultShape, GivensRotationMatrix, GramSchmidt, HankelMatrix, HermiteForm, HermitianTranspose, HessenbergForm, HilbertMatrix, HouseholderMatrix, IdentityMatrix, IntersectionBasis, IsDefinite, IsOrthogonal, IsSimilar, IsUnitary, JordanBlockMatrix, JordanForm, LA_Main, LUDecomposition, LeastSquares, LinearSolve, Map, Map2, MatrixAdd, MatrixExponential, MatrixFunction, MatrixInverse, MatrixMatrixMultiply, MatrixNorm, MatrixPower, MatrixScalarMultiply, MatrixVectorMultiply, MinimalPolynomial, Minor, Modular, Multiply, NoUserValue, Norm, Normalize, NullSpace, OuterProductMatrix, Permanent, Pivot, PopovForm, QRDecomposition, RandomMatrix, RandomVector, Rank, RationalCanonicalForm, ReducedRowEchelonForm, Row, RowDimension, RowOperation, RowSpace, ScalarMatrix, ScalarMultiply, ScalarVector, SchurForm, SingularValues, SmithForm, SubMatrix, SubVector, SumBasis, SylvesterMatrix, ToeplitzMatrix, Trace, Transpose, TridiagonalForm, UnitVector, VandermondeMatrix, VectorAdd, VectorAngle, VectorMatrixMultiply, VectorNorm, VectorScalarMultiply, ZeroMatrix, ZeroVector, Zip*] (4)

(b)

> $Multiply(A, B)$;

$$\begin{bmatrix} -3 & 1+3\,I & 3\,I \\ 4+3\,I & 9 & 6-2\,I \\ 6\,I & 6-2\,I & 6 \end{bmatrix} \tag{5}$$

(c)

> $Multiply(A, B) - Multiply(B, A)$;

$$\begin{bmatrix} -3 & 1+3\,I & 3\,I \\ 2+3\,I & 9 & 3-2\,I \\ -6+3\,I & 6+I & -6 \end{bmatrix} \tag{6}$$

(d)

> $Transpose(A)$;

$$\begin{bmatrix} -1 & 2 & 2\,I \\ 1 & 0 & -2\,I \\ I & 3 & 2 \end{bmatrix} \tag{7}$$

(f)

> $HermitianTranspose(A)$;

$$\begin{bmatrix} -1 & 2 & -2\,I \\ 1 & 0 & 2\,I \\ -I & 3 & 2 \end{bmatrix} \tag{8}$$

e)

> $HermitianTranspose(Transpose(A))$;

$$\begin{bmatrix} -1 & 1 & -I \\ 2 & 0 & 3 \\ -2\,I & 2\,I & 2 \end{bmatrix} \tag{9}$$

(g)

> $Determinant(B)$;

$$3 \tag{10}$$

(h)

> $Binverse := MatrixInverse(B)$;

$$Binverse := \begin{bmatrix} \frac{2}{3} & -I & \frac{1}{3}\,I \\ 0 & 1 & 0 \\ -\frac{1}{3}\,I & -2 & \frac{2}{3} \end{bmatrix} \tag{11}$$

> $Multiply(B, Binverse)$;

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12}$$

> $Determinant(A)$;

$$0 \tag{13}$$

Problem A9

A19

Eigenvalues

$$\det(\hat{M} - \lambda \hat{I}) = 0$$

$$\det\begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)^2 = 0 \qquad \lambda = 1$$

Eigenvectors

$$\begin{pmatrix} 1-1 & 1 \\ 0 & 1-1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

$$a_2 = 0$$

Eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\Rightarrow$ Eigenvectors don't span so diagonalizing matrix cannot be formed

25 (a) $T^\dagger = T = \tilde{T}^*$ ✓

(b)

$$\hat{T}\,|a\rangle = \lambda\,|a\rangle$$

$$= \lambda\,\hat{1}\,|a\rangle$$

$$(\hat{T} - \lambda\hat{1})\,|a\rangle = 0$$

Only if $\det(\hat{T} - \lambda\hat{1}) = 0$

$$\hat{T} - \lambda\hat{1} = \begin{pmatrix} 1-\lambda & 1-i \\ 1+i & -\lambda \end{pmatrix}$$

$$\det(\,) = -\lambda + \lambda^2 - 2 = 0$$

$\lambda = 2, -1$ Real

(c) $\lambda = 2$

$$\begin{pmatrix} -1 & 1-i \\ 1+i & -2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

$$-a_1 + a_2(1-i) = 0$$

Let $a_2 = 1$, $a_1 = 1-i$

$$|\lambda_2\rangle = (1-i, 1)$$

$$\langle \lambda_2 | \lambda_2 \rangle = 3$$

Normalize

$$|\lambda_2'\rangle = \frac{1}{\sqrt{3}}(1-i, 1)$$

$$= \frac{1}{\sqrt{3}} \begin{pmatrix} 1-i \\ 1 \\ 0 \end{pmatrix}$$

$\lambda = -1$

$$\begin{pmatrix} 2 & 1-i \\ 1+i & +1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

$$a_1(1+i) + a_2 = 0$$

If $a_1 = 1 \qquad a_2 = -1-i$

$$|\lambda_{-1}\rangle = \begin{pmatrix} 1 \\ -(1+i) \end{pmatrix}$$

$$|\lambda'_{-1}\rangle = \frac{|\lambda_{-1}\rangle}{\sqrt{\langle \lambda_{-1} | \lambda_{-1} \rangle}} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -(1+i) \end{pmatrix}$$

Check

$$\langle \lambda'_{-1} | \lambda_2 \rangle = \cancel{\frac{1}{3}\left(i-1 \,,\, 1 \right)}$$

$$= \frac{1}{3} \begin{pmatrix} 1 & i-1 \end{pmatrix} \begin{pmatrix} 1-i \\ 1 \end{pmatrix}$$

$$= 0 \checkmark$$

(26) (a)

$$Tr(\hat{T}) = \sum T_{ii} = 2+2+2 = 0$$

$$\det(\hat{T}) = \begin{vmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 2 & i \\ -i & 2 \end{vmatrix} - i\begin{vmatrix} -i & i \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} -i & 2 \\ 1 & -i \end{vmatrix}$$

$$= 2(4-1) - i(-2i-i) + (-1-2)$$

$$= 6 - 3 - 3 = 0$$

(b) Eigenvalues $\hat{T}$

$$\det(\hat{T} - \lambda I) = 0$$

$$\begin{vmatrix} 2-\lambda & i & 1 \\ -i & 2-\lambda & i \\ 1 & -i & 2-\lambda \end{vmatrix} = 0$$

$$\det(\hat{T} - \lambda I) = -9\lambda + 6\lambda^2 - \lambda^3 = 0 \quad \text{(Maple)}$$

$$= \lambda(\lambda^2 - 6\lambda + 9) = 0$$

$$= \lambda(\lambda - 3)^2 = 0$$

<u>Eigenvalues</u> $\lambda = 0, \quad \lambda = 3$ (degenerate)

$$\det(\hat{T}) = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 0 \quad \checkmark$$

$$\mathrm{Tr}(\hat{T}) = \lambda_1 + \lambda_2 + \lambda_3 = 6$$

<u>Diagonalized $\hat{T}$</u>

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

(c) Eigenvectors

$\underline{\lambda = 0}$

$$(\hat{T} - \lambda \hat{I})\vec{a} = 0 = \hat{T}\vec{a} \quad , \text{if } \lambda = 0$$

$$\begin{pmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} =$$

$$\begin{array}{ll} 2a_1 + ia_2 + a_3 = 0 & \text{first row} \\ a_1 - ia_2 + 2a_3 = 0 & \text{last row} \\ \hline 3a_1 + 3a_3 = 0 & \end{array}$$

If $a_1 = 1$, $a_3 = -1$, $\quad 2(1) + ia_2 + (-1) = 0$

$$a_2 = -1/i = i$$

$$|\lambda = 0\rangle = \begin{pmatrix} 1 \\ i \\ -1 \end{pmatrix} \qquad \langle 0|0\rangle = 1 + 1 + 1 = 3$$

$$\text{Normalized } |0\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ i \\ -1 \end{pmatrix}$$

Note, eigenvectors are undefined up to multiplication by a constant. The various solvers disagree on the eigenvectors because of a factor of -1.

Eigenvectors $\lambda = 3$

$$(\hat{T} - \lambda \hat{I})\vec{a} = \begin{pmatrix} 2-3 & i & 1 \\ -i & 2-3 & i \\ 1 & -i & 2-3 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

$$= \begin{pmatrix} -1 & i & 1 \\ -i & -1 & i \\ 1 & -i & -1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

$$-a_1 + i a_2 + a_3 = 0$$

$$-i a_1 - a_2 + i a_3 = 0 = i * \text{first row}$$

$$a_1 - i a_2 - a_3 = 0 = -1 \cdot \text{first row}$$

All three equations are redundant, so our eigenvectors must just satisfy one equation, so we can pick two coefficients

$a_1 = 1, \; a_2 = 0 \implies -1 + 0 + a_3 = 0 \implies a_3 = 1$

$a_1 = 1, \; a_3 = 0 \implies -1 + i a_2 + 0 = 0 \implies a_2 = -i$

Our two $\lambda = 3$ eigenvectors are

$$|3,1\rangle = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad |3,2\rangle = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

which are not orthogonal. Use Gram-Schmidt

$$|3,1\rangle' = \frac{|3,1\rangle}{\sqrt{\langle 3,1|3,1\rangle}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$ Normalized

Project out of $|3,2\rangle$

$$|b\rangle = |3,2\rangle - \langle 3,1|'3,2\rangle\, |3,1\rangle'$$

$$= \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} - \left[\frac{1}{\sqrt{2}}(1, 0, 1)\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}\right] \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} - \frac{1}{2}(1) \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} \\ -i \\ -1/2 \end{pmatrix}$$

Normalize

$$|3,2\rangle' = \frac{|b\rangle}{\sqrt{\langle b|b\rangle}} = \frac{1}{\sqrt{\frac{1}{4}+1+\frac{1}{4}}} \cdot \begin{pmatrix} \frac{1}{2} \\ -i \\ -1/2 \end{pmatrix}$$

$$= \frac{1}{\sqrt{\frac{6}{4}}} \begin{pmatrix} \frac{1}{2} \\ -i \\ -1/2 \end{pmatrix}$$

$$= \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2i \\ -1 \end{pmatrix}$$

Check Orthognol

$$\langle 3,2|3,1\rangle' = \frac{1}{\sqrt{6}} (1, 2i, -1) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$= 0$$

$$\langle \lambda=0|3,1\rangle' = \frac{1}{\sqrt{3}} (1, -i, -1) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$= 0$$

$$\langle \lambda=0|3,2\rangle' = \frac{1}{\sqrt{3}} (1, -i, -1) \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2i \\ -1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{18}} (1 - 2 + 1) = 0 \quad \checkmark$$

WolframAlpha computational knowledge engine

eigenvectors {{2, I, 1},{−I, 2, I}, {1, −I, 2}}

Input:

$$\text{Eigenvectors}\left[\begin{pmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{pmatrix}\right]$$

Result:

$v_1 \approx \{1, 0, 1\}$

$v_2 \approx \{i, 1, 0\}$

$v_3 \approx \{-1, -i, 1\}$

Corresponding eigenvalues:

$\lambda_1 = 3$

$\lambda_2 = 3$

$\lambda_3 = 0$

Problem A26

```
> with(LinearAlgebra);
```

[&x, Add, Adjoint, BackwardSubstitute, BandMatrix, Basis, BezoutMatrix, BidiagonalForm, BilinearForm, CharacteristicMatrix, CharacteristicPolynomial, Column, ColumnDimension, ColumnOperation, ColumnSpace, CompanionMatrix, ConditionNumber, ConstantMatrix, ConstantVector, Copy, CreatePermutation, CrossProduct, DeleteColumn, DeleteRow, Determinant, Diagonal, DiagonalMatrix, Dimension, Dimensions, DotProduct, EigenConditionNumbers, Eigenvalues, Eigenvectors, Equal, ForwardSubstitute, FrobeniusForm, GaussianElimination, GenerateEquations, GenerateMatrix, GetResultDataType, GetResultShape, GivensRotationMatrix, GramSchmidt, HankelMatrix, HermiteForm, HermitianTranspose, HessenbergForm, HilbertMatrix, HouseholderMatrix, IdentityMatrix, IntersectionBasis, IsDefinite, IsOrthogonal, IsSimilar, IsUnitary, JordanBlockMatrix, JordanForm, LA_Main, LUDecomposition, LeastSquares, LinearSolve, Map, Map2, MatrixAdd, MatrixExponential, MatrixFunction, MatrixInverse, MatrixMatrixMultiply, MatrixNorm, MatrixPower, MatrixScalarMultiply, MatrixVectorMultiply, MinimalPolynomial, Minor, Modular, Multiply, NoUserValue, Norm, Normalize, NullSpace, OuterProductMatrix, Permanent, Pivot, PopovForm, QRDecomposition, RandomMatrix, RandomVector, Rank, RationalCanonicalForm, ReducedRowEchelonForm, Row, RowDimension, RowOperation, RowSpace, ScalarMatrix, ScalarMultiply, ScalarVector, SchurForm, SingularValues, SmithForm, SubMatrix, SubVector, SumBasis, SylvesterMatrix, ToeplitzMatrix, Trace, Transpose, TridiagonalForm, UnitVector, VandermondeMatrix, VectorAdd, VectorAngle, VectorMatrixMultiply, VectorNorm, VectorScalarMultiply, ZeroMatrix, ZeroVector, Zip]

```
> T:=<<2 | I | 1> , <-I | 2 | I> , <1 | -I | 2>>;
```

$$T := \begin{bmatrix} 2 & I & 1 \\ -I & 2 & I \\ 1 & -I & 2 \end{bmatrix}$$

```
> Trace(T);
```

$$6$$

```
> Determinant(T);
```

$$0$$

```
> T2:=<<2-lambda | I | 1> , <-I | 2-lambda | I> , <1 | -I |
  2-lambda>>;
```

$$T2 := \begin{bmatrix} 2-\lambda & I & 1 \\ -I & 2-\lambda & I \\ 1 & -I & 2-\lambda \end{bmatrix}$$

```
> Determinant(T2);
```

$$-9\,\lambda + 6\,\lambda^2 - \lambda^3$$

```
> Eigenvalues(T);
```

$$\begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$$

```
> Eigenvectors(T);
```

$$\begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 & 1 & I \\ -I & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

```
> a3:=<1, -2*I, -1>;
```

$$a3 := \begin{bmatrix} 1 \\ -2\,I \\ -1 \end{bmatrix}$$

```
> T.a3;
```

$$\begin{bmatrix} 3 \\ -6\,I \\ -3 \end{bmatrix}$$

Note, the above shows a3 is an eigenvector with lambda=3