PHYS 4073 - Quantum Mechanics- Homework Set 5

Reading Assignment: Chapter 3

Due at the beginning of class Friday October 15th.

Griffiths' Problems

3.13

3.17

3.22

3.23

3.25

3.37

Additional Problems

A1 Compute the uncertainty relation between the kinetic energy and the potential energy for the simple harmonic oscillator.

A2 Find the differential equation for the time evolution of the expectation value of the kinetic energy for the simple harmonic oscillator.

A3 Show that the classical equations of motion are recovered for the Simple Harmonic Oscillator(SHO) by examining the time evolution of the average position and momentum.

A4 Using the normalized raising and lowering operators, verify that the uncertainty principle is satisfied for the ground state of the SHO.

A5 Using the normalized raising and lowering operators, write the matrices representing the $\hat{X}$ and $\hat{P}$ operators in the energy basis.

25pts each

3.13

(a) $[AB, C] = ABC - CAB$

$A[B,C] + [A,C]B$

$= ABC \underbrace{- ACB + ACB}_{0} - \cancel{ABC}\ CAB$

$= ABC - CAB$ ✓

(b) $[x^n, p] = x^n p - p x^n$

Look at the second term and commute once,

$[x,p] = xp - px = i\hbar$

$p x^n = x p x^{n-1} - i\hbar x^{n-1}$

Each time we commute p we pick up an $-i\hbar x^{n-1}$

so eventually

$p x^n = \cancel{p} x^n p - i n \hbar x^{n-1}$

and $[x^n, p] = x^n p - (x^n p - in\hbar x^{n-1})$

$$= in\hbar x^{n-1}$$

(c) Write $f(x)$ as a power series

$$f(x) = \sum a_n x^n$$

$$[f(x), p] = \sum a_n [x^n, p]$$

$$= \sum i n a_n x^{n-1} \hbar$$

$$= i\hbar \frac{df}{dx}$$

3.17

$$\frac{d}{dt}\langle a\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{A}]\rangle + \left\langle\frac{\partial \hat{A}}{\partial t}\right\rangle$$

In all cases in this problem, the second term is zero.

(a) $\hat{A} = \hat{1}$

$$\frac{d}{dt}\langle\hat{1}\rangle = \frac{d}{dt}\langle\psi|\psi\rangle = \frac{i}{\hbar}\langle\underbrace{[\hat{H},\hat{1}]}_{0}\rangle$$

$$\Rightarrow \frac{d}{dt}\langle\psi|\psi\rangle = 0$$

$\Rightarrow$ Conservation of probability or particles.

(b) $\hat{A} = \hat{H}$, $[\hat{H},\hat{H}] = 0$

$$\frac{d}{dt}\langle\hat{H}\rangle = \frac{d}{dt}\langle E\rangle = 0$$

$\Rightarrow$ Energy is conserved.

(c) $[\hat{H}, \hat{x}] = \left[\frac{\hat{p}^2}{2m} + V(\hat{x}), \hat{x} \right]$

$$= \left[\frac{\hat{p}^2}{2m}, \hat{x} \right] = -\frac{1}{2m} [\hat{x}, \hat{p}^2]$$

$$= -\frac{1}{2m} i\hbar \frac{d\hat{p}^2}{d\hat{p}} = -\frac{i\hbar}{m} \hat{p}$$

$$\frac{d}{dt} \langle \hat{x} \rangle = \frac{i}{\hbar} \langle [\hat{H}, \hat{x}] \rangle$$

$$= \frac{\langle \hat{p} \rangle}{m}$$ ✓

(d) $[\hat{H}, \hat{p}] = \left[\frac{\hat{p}^2}{2m} + V(\hat{x}), \hat{p} \right]$

$$= [V(\hat{x}), \hat{p}] = i\hbar \frac{dV}{dx}$$

$$\frac{d}{dt} \langle \hat{p} \rangle = \frac{i}{\hbar} \langle [\hat{H}, \hat{p}] \rangle = \frac{i}{\hbar} \left\langle \left(i\hbar \frac{dV}{dx} \right) \right\rangle$$

$$= -\left\langle \frac{dV}{dx} \right\rangle$$ = force ✓

3.22

(a) $\langle\alpha| = -i\langle 1| - 2\langle 2| + i\langle 3|$

$\langle\beta| = -i\langle 1| + 2\langle 3|$

(b) $\langle\alpha|\beta\rangle = \cancel{1-4=-3}$

$= 1 + 2i$

$\langle\beta|\alpha\rangle = 1 - 2i = \langle\alpha|\beta\rangle^*$

(c)

$$\hat{A} = \begin{pmatrix} \langle 1|\alpha\rangle\langle\beta|1\rangle & \langle 1|\alpha\rangle\langle\beta|2\rangle & \langle 1|\alpha\rangle\langle\beta|3\rangle \\ \langle 2|\alpha\rangle\langle\beta|1\rangle & \langle 2|\alpha\rangle\langle\beta|2\rangle & \langle 2|\alpha\rangle\langle\beta|3\rangle \\ \langle 3|\alpha\rangle\langle\beta|1\rangle & \langle 3|\alpha\rangle\langle\beta|2\rangle & \langle 3|\alpha\rangle\langle\beta|3\rangle \end{pmatrix}$$

$$\hat{A} = \begin{pmatrix} i(-i) & i\,0 & i\,2 \\ -2(-i) & -2\cdot 0 & (-2)(2) \\ (-i)(-i) & (-i)(0) & (-i)(2) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 2i \\ 2i & 0 & -4 \\ -1 & 0 & -2i \end{pmatrix}$$

Not Hermitian as we would expect since

$$\hat{A}^{+} = |\beta\rangle\langle\alpha| \neq \hat{A} = |\alpha\rangle\langle\beta|$$

3.23

$$\hat{H} = \begin{pmatrix} \langle 1|H|1\rangle & \langle 1|H|2\rangle \\ \langle 2|H|1\rangle & \langle 2|H|2\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \varepsilon & \varepsilon \\ \varepsilon & -\varepsilon \end{pmatrix}$$

<u>Eigenvalues</u>

$$\det(\hat{H} - \lambda \hat{I}) = 0 = \det \begin{pmatrix} \varepsilon - \lambda & \varepsilon \\ \varepsilon & -\varepsilon - \lambda \end{pmatrix}$$

$$= (\varepsilon - \lambda)(-\varepsilon - \lambda) - \varepsilon^2 = 0$$

$$= -\varepsilon^2 + \lambda^2 - \varepsilon^2 = 0$$

$$\lambda = \pm \varepsilon\sqrt{2} = \text{energy values.}$$

Eigenvectors

$\lambda = \varepsilon\sqrt{2}$

$$\begin{pmatrix} \varepsilon - \varepsilon\sqrt{2} & \varepsilon \\ \varepsilon & -\varepsilon - \varepsilon\sqrt{2} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

$$a_1(1-\sqrt{2}) + a_2 = 0$$

If $a_1 = 1$ $\quad a_2 = -(1-\sqrt{2})$

$$|\varepsilon\sqrt{2}\rangle = |1\rangle - (1-\sqrt{2})|2\rangle$$

* The problem did not require normalization.

$\lambda = -\varepsilon\sqrt{2}$

$$\begin{pmatrix} \varepsilon + \varepsilon\sqrt{2} & \varepsilon \\ \varepsilon & -\varepsilon + \varepsilon\sqrt{2} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

$$a_1(\varepsilon + \varepsilon\sqrt{2}) + \varepsilon a_2 = 0$$

If $a_1 = 1$, $a_2 = -(1+\sqrt{2})$

$$|-\varepsilon\sqrt{2}\rangle = |1\rangle - (1+\sqrt{2})|2\rangle$$

With respect to the new basis

$$\hat{H} = \begin{pmatrix} \varepsilon\sqrt{2} & 0 \\ 0 & -\varepsilon\sqrt{2} \end{pmatrix}$$

3.25 $\quad |a_1\rangle = 1 \qquad |a_2\rangle = x$

$$|a_3\rangle = x^2 \qquad |a_4\rangle = x^3$$

$$\langle a|b\rangle = \int_{-1}^{1} a^* b \, dx$$

$$\langle a_1|a_1\rangle = \int_{-1}^{1} dx = 2$$

$$\boxed{|e_1\rangle = \frac{|a_1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}}$$

$$|b_2\rangle = |a_2\rangle - \langle e_1|a_2\rangle |e_1\rangle$$

$$\langle e_1|a_2\rangle = \frac{1}{\sqrt{2}} \int_{-1}^{1} x \, dx = 0$$

$$|b_2\rangle = |a_2\rangle$$

$$\langle b_2|b_2\rangle = \int_{-1}^{1} x^2 dx = \left.\frac{x^3}{3}\right|_{-1}^{1} = 2/3$$

$$\boxed{|e_2\rangle = \frac{|b_2\rangle}{\sqrt{2/3}} = \sqrt{\frac{3}{2}}\, x}$$

$$\langle e_1 | a_3 \rangle = \frac{1}{\sqrt{2}} \int_{-1}^{1} x^2 dx = \frac{1}{\sqrt{2}} \frac{2}{3}$$

$$\langle e_2 | a_3 \rangle = \sqrt{\frac{3}{2}} \int_{-1}^{1} x^3 dx = 0$$

$$| b_3 \rangle = | a_3 \rangle - \langle e_1 | a_3 \rangle | e_1 \rangle - \langle e_2 | a_3 \rangle | e_2 \rangle$$

$$= x^2 - \frac{1}{\sqrt{2}} \frac{2}{3} | e_1 \rangle$$

$$= x^2 - \frac{1}{3}$$

$$\langle b_3 | b_3 \rangle = \int_{-1}^{1} \left(x^2 - \frac{1}{3} \right)^2 dx$$

$$= \int_{-1}^{1} x^4 - \frac{2}{3} x^2 + \frac{1}{9} \, dx$$

$$= \frac{x^5}{5} - \frac{2x^3}{9} + \frac{x}{9} \Big|_{-1}^{1}$$

$$= \frac{2}{5} - \frac{4}{9} + \frac{2}{9}$$

$$= \frac{8}{45}$$

$$|e_3\rangle = \frac{|b_3\rangle}{\sqrt{8/45}}$$

$$= \sqrt{\frac{5}{2}} \cdot \frac{3}{2}\left(x^2 - \frac{1}{3}\right)$$

$$\boxed{|e_3\rangle = \sqrt{\frac{5}{2}}\left(\frac{3}{2}x^2 - \frac{1}{2}\right)}$$

$\langle e_1|a_4\rangle = 0$ $\quad\quad$ $\langle e_3|a_4\rangle = 0$

odd function / even range.

$$\langle e_2|a_4\rangle = \sqrt{\frac{3}{2}}\int_{-1}^{1} x^4\,dx = \frac{2}{5}\sqrt{3/2}$$

$$|b_4\rangle = |a_4\rangle - \langle e_2|a_4\rangle|e_2\rangle$$

$$= x^3 - \left(\frac{2}{5}\sqrt{3/2}\right)\sqrt{\frac{3}{2}}\,x$$

$$= x^3 - \frac{3}{5}x$$

$$\langle b_4 | b_4 \rangle = \int_{-1}^{1} \left(x^3 - \tfrac{3}{5} x \right)^2 dx$$

$$= \int_{-1}^{1} \left(x^6 - \frac{6}{5} x^4 + \frac{9}{25} x^2 \right) dx$$

$$= \frac{x^7}{7} - \frac{6}{25} x^5 + \frac{3}{25} x^3 \Big|_{-1}^{1}$$

$$= \frac{2}{7} - \frac{12}{25} + \frac{6}{25}$$

$$= \frac{2}{7} - \frac{6}{25}$$

$$= \frac{50}{175} - \frac{42}{175} = \frac{8}{175}$$

$$|e_4\rangle = \frac{|b_4\rangle}{\sqrt{8/175}} = \frac{5}{2} \sqrt{\frac{7}{2}} \left(x^3 - \frac{3}{5} x \right)$$

$$= \sqrt{\frac{7}{2}} \left(\frac{5}{2} x^3 - \frac{3}{2} x \right)$$

(3.37) Find the eigenvalues and vectors.

$$\det\left(\hat{H}-\lambda\hat{1}\right)=\det\begin{pmatrix} a-\lambda & 0 & b \\ 0 & c-\lambda & 0 \\ b & 0 & a-\lambda \end{pmatrix}=0$$

$$\cancel{(a-\lambda)(c)(a-\lambda) + b(0-bc) = 0}$$

~~c=0 eigenvalue~~

$$0 = (a-\lambda)(c-\lambda)(a-\lambda) + b\left(-b(c-\lambda)\right) = 0$$

$$\lambda = c$$

$$(a-\lambda)^2 - b^2 = 0$$

$$a-\lambda = \pm b$$

$$\lambda = a-b,\ a+b$$

Eigenvectors

$\lambda = c$ $\qquad |c\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda = a + b$

$$\begin{pmatrix} -b & 0 & b \\ 0 & c-(a+b) & 0 \\ b & 0 & -b \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

$$-ba_1 + ba_3 = 0$$

$$a_1 = 1 \qquad a_3 = 1$$

$$|a+b\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$\lambda = a - b$

$$\begin{pmatrix} b & 0 & b \\ 0 & c-(a-b) & 0 \\ b & 0 & b \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

$$ba_1 + ba_3 = 0 \qquad a_1 = 1 \qquad a_3 = -1$$

$$|a-b\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

(a) $\quad |S(0)\rangle = |c\rangle \quad$ stationary state

$$|S(t)\rangle = e^{-ic/\hbar\, t}\,|S(0)\rangle$$

(b) $$|S(0)\rangle = \frac{\sqrt{2}}{2}\Big(|a+b\rangle - |a-b\rangle\Big)$$

$$|S(t)\rangle = \frac{1}{\sqrt{2}}\Big(e^{-i(a+b)t/\hbar}\,|a+b\rangle - e^{-i(a-b)t/\hbar}\,|a-b\rangle\Big)$$

Let $\omega_+ = \dfrac{a+b}{\hbar} \qquad \omega_- = \dfrac{a-b}{\hbar}$

$$|S(t)\rangle = \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\omega_+ t} \\ 0 \\ e^{-i\omega_+ t} \end{pmatrix} - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\omega_- t} \\ 0 \\ -e^{-i\omega_- t} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} e^{-i\omega_+ t} - e^{-i\omega_- t} \\ 0 \\ e^{-i\omega_+ t} + e^{-i\omega_- t} \end{pmatrix}$$

$$|s(t)\rangle = \frac{1}{2}\begin{pmatrix} e^{-i(a+b)t/\hbar} - e^{-i(a-b)t/\hbar} \\ 0 \\ e^{-i(a+b)t/\hbar} + e^{-i(a-b)t/\hbar} \end{pmatrix}$$

$$= \frac{1}{2} e^{-iat/\hbar}\begin{pmatrix} e^{-ibt/\hbar} - e^{+ibt/\hbar} \\ 0 \\ e^{-ibt/\hbar} + e^{ibt/\hbar} \end{pmatrix}$$

$$= e^{-iat/\hbar}\begin{pmatrix} i \sin bt/\hbar \\ \\ \cos bt/\hbar \end{pmatrix}$$