A1

Uncertainty Relations

Kinetic Energy SHO $\qquad \hat{T} = \dfrac{\hat{P}^2}{2m}$

Potential Energy $\qquad \hat{V} = \frac{1}{2} m\omega^2 \hat{X}^2$

Uncertainty Relation

$$\sigma_T^2 \sigma_V^2 \geq \left(\frac{1}{2i} \langle \psi | [\hat{T}, \hat{V}] | \psi \rangle \right)^2$$

$$[\hat{T}, \hat{V}] = \left[\frac{\hat{P}^2}{2m}, \frac{1}{2} m\omega^2 \hat{X}^2 \right]$$

$$= \frac{1}{4} \omega^2 [\hat{P}^2, \hat{X}^2]$$

Using 3.13 results,

$$[\hat{P}^2, \hat{X}^2] = \hat{P} \underbrace{[\hat{P}, \hat{X}^2]}_{-2i\hbar\hat{X}} + [\hat{P}, \hat{X}^2] \hat{P}$$

$$= -2i\hbar \left(\hat{P}\hat{X} + \hat{X}\hat{P} \right)$$

$\cancel{\sigma_T^2 \sigma_V^2 \geq}$

$$[\hat{T}, \hat{V}] = \frac{-i\hbar\omega^2}{2}(\hat{p}\hat{x} + \hat{x}\hat{p})$$

$$\sigma_T^2 \sigma_{\cancel{x}}^2 \geq \left(\frac{1}{2i} \cdot \left(\frac{-i\hbar\omega^2}{2}\right)\langle \hat{p}\hat{x} + \hat{x}\hat{p}\rangle\right)^2$$

$$\frac{\hbar^2\omega^4}{16}\langle \hat{p}\hat{x} + \hat{x}\hat{p}\rangle^2$$

$$[\hbar] = J \cdot s = N m s = kg \frac{m^2}{s} = [p][x] \checkmark$$

(A2) $T = P^2/2m$

$$\frac{d\langle T\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H},\hat{T}]\rangle + \underbrace{\left\langle \frac{\partial \hat{T}}{\partial t}\right\rangle}_{=0}$$

$$[\hat{H},\hat{T}] = [\hat{T}+\hat{V},\hat{T}] = [\hat{V},\hat{T}]$$

$$= \frac{i\hbar\omega^2}{2}(\hat{P}\hat{X}+\hat{X}\hat{P})$$

$$\frac{d\langle T\rangle}{dt} = \left(\frac{i}{\hbar}\right)\left(\frac{i\hbar\omega^2}{2}\right)\langle\hat{P}\hat{X}+\hat{X}\hat{P}\rangle$$

$$= -\frac{\omega^2}{2}\langle\hat{P}\hat{X}+\hat{X}\hat{P}\rangle$$

A3

$$\hat{H} = \frac{\hat{P}^2}{2m} + \frac{1}{2} m\omega^2 \hat{X}^2$$

$$= \frac{\hat{P}^2}{2m} + \frac{1}{2} k \hat{X}^2$$

$$\frac{d}{dt}\langle x\rangle = \frac{i}{\hbar}\langle [\hat{H}, \hat{X}]\rangle$$

$$= \frac{i}{2m\hbar}\langle \underbrace{[\hat{P}^2, \hat{X}]}_{-i\hbar \cdot \frac{dP^2}{dP} \, = \, -2i\hbar \hat{P}}\rangle$$

3.13

$$= \frac{i}{2m\hbar}\langle -2i\hbar \hat{P}\rangle$$

$$= \frac{\langle \hat{P}\rangle}{m} \checkmark$$

$$\frac{d}{dt}\langle p\rangle = \frac{i}{\hbar}\langle [\hat{H}, \hat{P}]\rangle$$

3.13

$$= \frac{ik}{2\hbar}\langle [\hat{X}^2, \hat{P}]\rangle = \frac{ik}{2\hbar}\langle 2i\hbar \hat{X}\rangle$$

$$= -k\langle \hat{X}\rangle \checkmark$$ Newton II

= force.

A4 From notes, normalized ladder operators

$$\hat{a}_+|n\rangle = \sqrt{n+1}\ |n+1\rangle$$

$$\hat{a}_-|n\rangle = \sqrt{n}\ |n-1\rangle$$

with $\hat{H}|n\rangle = E_n|n\rangle$,

$$\hat{X} = \sqrt{\frac{\hbar}{2m\omega}}\ (\hat{a}_+ + \hat{a}_-)$$

$$\hat{X}^2 = \frac{\hbar}{2m\omega}\ (\hat{a}_+^2 + \hat{a}_-^2 + \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+)$$

$$\hat{P} = i\sqrt{\frac{\hbar m\omega}{2}}\ (\hat{a}_+ - \hat{a}_-)$$

$$\hat{P}^2 = -\frac{\hbar m\omega}{2}\ (\hat{a}_+^2 + \hat{a}_-^2 - \hat{a}_+\hat{a}_- - \hat{a}_-\hat{a}_+)$$

What do ladder operators do to ground state?

$$\hat{a}_-|0\rangle = 0$$

$$\hat{a}_+|0\rangle = \sqrt{0+1}\ |0+1\rangle = |1\rangle$$

Expectation Values

$$\langle x \rangle = \langle 0|\hat{x}|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,\langle 0|\hat{a}_+ + \hat{a}_-|0\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left[\underbrace{\langle 0|\hat{a}_+|0\rangle}_{=\langle 0|1\rangle = 0} + \underbrace{\langle 0|\hat{a}_-|0\rangle}_{=0}\right.$$

$$= 0$$

Likewise because it contains a single raising or lowering operator, $\langle p \rangle = 0$. We need two operators to bring $|0\rangle$ back to $|0\rangle$ so $\langle 0|$ doesn't give zero

---

$$\langle x^2 \rangle = \frac{\hbar}{2m\omega}\langle 0|\underbrace{a_+^2}_{=0} + \underbrace{a_-^2}_{=0} + \underbrace{\hat{a}_+\hat{a}_-}_{\hat{a}_-|0\rangle = 0} + \hat{a}_-\hat{a}_+|0\rangle$$

final state $|2\rangle$

$$= \frac{\hbar}{2m\omega}\langle 0|\hat{a}_-\hat{a}_+|0\rangle = \frac{\hbar}{2m\omega}\langle 0|\hat{a}_-|1\rangle$$

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}\langle 0|0\rangle = \frac{\hbar}{2m\omega}$$

since $\hat{a}_-|1\rangle = \sqrt{1}\,|0\rangle = |0\rangle$

$$\langle p^2\rangle = -\frac{\hbar m\omega}{2}\langle 0|\,a_+^2 + a_-^2 - \hat{a}_+\hat{a}_- - \hat{a}_-\hat{a}_+\,|0\rangle$$

$$= -\frac{\hbar m\omega}{2}\langle 0| -\hat{a}_-\hat{a}_+|0\rangle$$

$$= \frac{\hbar m\omega}{2}$$

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{\hbar}{2m\omega}}$$

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\frac{\hbar m\omega}{2}}$$

$$\sigma_x\sigma_p = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{\hbar m\omega}{2}} = \frac{\hbar}{2} \quad \checkmark$$

minimum uncertainty since ground state is Gaussian.

A5

$$\hat{H}|n\rangle = E_n|n\rangle$$

$$X_{ij} = \langle i|\hat{x}|j\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\langle i|\hat{a}_+ + \hat{a}_-|j\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left[\langle i|\hat{a}_+|j\rangle + \langle i|\hat{a}_-|j\rangle\right]$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{j+1}\langle i|j+1\rangle + \sqrt{j}\langle i|j-1\rangle\right.$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{j+1}\,\delta_{i,j+1} + \sqrt{j}\,\delta_{i,j-1}\right)$$

$$\hat{X} = \sqrt{\frac{\hbar}{2m\omega}}\begin{pmatrix} 0 & \sqrt{2} & 0 & \cdots & & \\ \sqrt{2} & 0 & \sqrt{3} & 0 & \cdots & \\ 0 & \sqrt{3} & 0 & 2 & 0 & \cdots \\ 0 & 0 & 2 & \ddots & & \\ \vdots & \vdots & & & \ddots & \end{pmatrix}$$

$$P_{ij} = \langle i | \hat{p} | j \rangle = i\sqrt{\frac{\hbar m\omega}{2}}\, \langle i | \hat{a}_+ - \hat{a}_- | j \rangle$$

$$= i\sqrt{\frac{\hbar m\omega}{2}} \left[\langle i | a_+ | j \rangle - \langle i | a_- | j \rangle \right]$$

$$i\sqrt{\frac{\hbar m\omega}{2}} \left(\sqrt{j+1}\, \delta_{i,j+1} - \sqrt{j}\, \delta_{i,j-1} \right)$$

$$\hat{P} = \begin{pmatrix} 0 & -\sqrt{2} & 0 & 0 & 0 \\ \sqrt{2} & 0 & -\sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 & 2 & \\ 0 & 0 & 2 & 0 & \\ \vdots & & & & \end{pmatrix} i\sqrt{\frac{\hbar m\omega}{2}}$$