# PHYS 4073 - Quantum Mechanics- Homework Set 6

**Reading Assignment: No additional material.**

Due at 5:45pm Monday October 25th in my box or at my office.

## Griffiths' Problems

Work four of the seven following Griffiths' problems

3.4

3.10

3.14

3.27

3.30

## Additional Problems

All additional problems use the following two matrices where $\hat{H}$ is the hamiltonian and $\hat{A}$ is a second matrix associated with physical quantity $a$:

$$\hat{H} = \hbar\omega \begin{pmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\hat{A} = \hbar \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 1 \end{pmatrix}$$

**A1** Are either or both $\hat{A}$ and $\hat{H}$ Hermitian? Why? What does this imply?

**A2** Calculate the uncertain relation, $\sigma_E^2 \sigma_a^2 >?$ for a system in the energy ground state.

**A3** A system is prepared in the energy ground state. What values of $a$ could be observed with what probability?

**A4** A system is prepared in a linear combination of the energy ground state and the first excited state:

$$|\psi> = \frac{1}{\sqrt{2}}|\phi_0> + \frac{1}{\sqrt{2}}|\phi_1>$$

Calculate the expectation value of $a$, $< a >$, as a function of time directly by calculating $|\psi(t)>$ and then calculating the expectation value.

**A5** A measurement is performed that finds the system is in a state with the lowest value of $a$. Calculate the expectation value of the energy for this state.

Each 25pts

3.4 If $I(\hat{A}|b\rangle, |a\rangle) = I(|b\rangle, \hat{A}|a\rangle)$

and $I(\hat{B}|b\rangle, |a\rangle) = I(|b\rangle, \hat{B}|a\rangle)$

then $I(|b\rangle, (\hat{A}+\hat{B})|a\rangle) = I(|b\rangle, \hat{A}|a\rangle)$

$+ I(|b\rangle, \hat{B}|a\rangle)$

$= I(\hat{A}|b\rangle, |a\rangle) + I(\hat{B}|b\rangle, |a\rangle)$

$= I((\hat{A}+\hat{B})|b\rangle, a\rangle$

$\Rightarrow (\hat{A}+\hat{B})^{+} = (\hat{A}+\hat{B})$

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(b) $(\alpha\hat{Q})^{+} = \alpha^{*}\hat{Q}^{+} = \alpha^{*}\hat{Q}$

$\Rightarrow$ $\alpha$ real for Hermitian

(c) $(\hat{A}\hat{B})^{+} = \hat{B}^{+}\hat{A}^{+} = \hat{B}\hat{A}$

Hermitian if they commute

(d) The inner product for these operators is $\langle f | g \rangle = \int f^* g \, dx$

<u>Position</u> $\langle f | x g \rangle = \int f^* x g \, dx$

$$= \int x f^* g \, dx$$

$$= \langle x f | g \rangle \equiv I(xf, g)$$

<u>Hamiltonian</u>

Since $V(x)$ is just a function of $x$, the above reasoning shows it is Hermitian.

$$I\left(f, -\frac{\hbar^2}{2m}\frac{d^2 g}{dx^2}\right) = -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} f^* \frac{d^2 g}{dx^2} \, dx$$

$$I\left(f, \frac{d^2 g}{dx^2}\right) = \int_{-\infty}^{\infty} f^* \frac{d^2 g}{dx^2} \, dx$$

Integrate by parts

$$\int f^* \frac{d^2 g}{dx^2}\, dx = f^* \frac{dg}{dx}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{df^*}{dx} \frac{dg}{dx}\, dx$$

$$= f^* \frac{dg}{dx}\Big|_{-\infty}^{\infty} - g \frac{df^*}{dx}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} g \frac{d^2 f^*}{dx^2}\, dx$$

$$= I\left( \frac{d^2 f}{dx^2}, g \right)$$

We may assume surface terms vanish for square-integrable functions.

3.10

$$\phi_1 = \sqrt{\frac{2}{a}} \sin \frac{\pi x}{a}$$

$$\hat{P}_x = \frac{\hbar}{i} \frac{\partial}{\partial x}$$

$$\hat{P}_x \phi_1 = \left(\frac{\hbar}{i}\right) \sqrt{\frac{2}{a}} \frac{\pi}{a} \cos \frac{\pi x}{a}$$

$$\neq p\phi_1$$

<u>Why?</u> The ground state is formed by a left and right traveling wave.

3.27

a) State collapses to $\psi_1$

b) $\psi_1 = \frac{3}{5}\phi_1 + \frac{4}{5}\phi_2$ (Normalized)

<u>Outcomes</u>

$b_1 \qquad P(b_1) = \left(\frac{3}{5}\right)^2 = \frac{9}{25}$

$b_2 \qquad P(b_2) = \left(\frac{4}{5}\right)^2 = \frac{16}{25}$

c) If $b_1$ was observed, then state ~~was~~ is $\phi_1$

$\cancel{4\psi_1 + 3\psi_2 = \frac{24}{5}\phi_1}$

$\cancel{4\psi_1 - 3\psi_2 =}$

$\cancel{\phi_1 = \frac{5}{6}\psi_1 +}$

$$3\psi_1 + 4\psi_2 = \frac{25}{5}\phi_1$$

$$\phi_1 = \frac{3}{5}\psi_1 + \frac{4}{5}\psi_2$$

So the outcomes are

$a_1$ $\quad P(a_1) = \frac{9}{25}$

$a_2$ $\quad P(a_2) = \frac{16}{25}$

If $b_2$ were observed, the state becomes $\phi_2$

$$4\psi_1 - 3\psi_2 = \frac{25}{5}\phi_2$$

$$\phi_2 = \frac{4}{5}\psi_1 - \frac{3}{5}\psi_2$$

<u>outcomes</u>

$a_1$ : $P(a_1) = \frac{16}{25}$

$a_2$ : $P(a_2) = \frac{9}{25}$

Total probability of observing $a_1$ after SOME measurement of b

$$P = \underbrace{\frac{9}{25}}_{b_1} \cdot \frac{9}{25} + \underbrace{\frac{16}{25}}_{b_2} \cdot \frac{16}{25}$$

$$= 0.54$$

3.30

(a) Normalize $\qquad \psi = \dfrac{A}{x^2+a^2}$

$$1 = \int_{-\infty}^{\infty} \frac{A^2}{(x^2+a^2)^2}\,dx = A^2 \frac{\pi}{2a^3}$$

$$A = \sqrt{\frac{2a^3}{\pi}}$$

(b) $\langle x \rangle = 0 \qquad$ (~~odd fn~~ even function / odd range)

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 \psi^* \psi\, dx = \int_{-\infty}^{\infty} \frac{A^2 x^2\, dx}{(x^2+a^2)^2}$$

$$= a^2$$

$$\cancel{\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = a}$$

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = a$$

(c) $$\Phi(p,0) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \frac{A}{x^2+a^2} e^{-ipx/\hbar}\, dx$$

$$= \frac{A}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \frac{\cos px/\hbar - i \sin px/\hbar}{x^2+a^2}\, dx$$

The sin piece is zero because of odd function / even range.

$$\Phi(p,0) = \frac{2A}{\sqrt{2\pi\hbar}} \int_0^{\infty} \frac{\cos px/\hbar}{x^2+a^2}\, dx$$

$$=$$

Schaum's

$$\int_0^\infty \frac{\cos mx}{x^2+a^2}\,dx = \frac{\pi}{2a} e^{-ma}$$

$$\Phi(p,0) = \frac{2A}{\sqrt{2\pi\hbar}} \int_0^\infty \frac{\cos px/\hbar}{x^2+a^2}\,dx$$

$$= \frac{2A}{\sqrt{2\pi\hbar}} \frac{\pi}{2a} e^{-pa/\hbar}$$

$$= \frac{2}{\sqrt{2\pi\hbar}} \frac{\pi}{2a} \sqrt{\frac{2a^3}{\pi}}\, e^{-pa/\hbar}$$

$$= \sqrt{\frac{a}{\hbar}}\, e^{-pa/\hbar}$$

Or if $p<0$,

$$\Phi(p,0) = \sqrt{\frac{a}{\hbar}}\, e^{-|p|a/\hbar}$$

Maple gives

$$+\sqrt{\frac{a}{\hbar}}\left(\cosh\frac{pa}{\hbar} - \sinh\frac{pa}{\hbar}\right)$$

$$= \sqrt{\frac{a}{\hbar}}\, e^{-pa/\hbar}$$

Clearly not correct for $p \to -\infty$

$$\Phi(p,0) = \sqrt{\frac{a}{\hbar}}\, e^{-|p|a/\hbar}$$

Check normalization

$$1 = \frac{a}{\hbar}\int_{-\infty}^{\infty} e^{-2|p|a/\hbar}\, dp$$

$$= \frac{2a}{\hbar}\int_{0}^{\infty} e^{-2pa/\hbar}\, dp$$

$$= 1$$

(d) $\langle p \rangle = 0$ odd function even range

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} p^2 \Phi^* \Phi \, dp$$

$$= \frac{2a}{\hbar} \int_0^{\infty} p^2 e^{-2pa/\hbar} \, dp$$

$$= \frac{2a}{\hbar} \cdot \left(-\frac{1}{4}\right)\left(-\frac{\hbar^3}{a^3}\right)$$

$$= \frac{\hbar^2}{2a^2}$$

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \frac{1}{\sqrt{2}} \frac{\hbar}{a}$$

(e) Uncertainty

$$\sigma_x \sigma_p = a \cdot \frac{1}{\sqrt{2}} \frac{\hbar}{a} = \frac{1}{\sqrt{2}} \hbar = 0.707 \hbar > 0.5 \hbar$$

✓

$> int\left(\frac{1}{(x^2+a^2)^2}, x=-infinity..infinity\right);$

$$\begin{cases} -\frac{1}{2}\frac{\pi}{a^3} & a<0 \\ \infty & a=0 \\ \frac{1}{2}\frac{\pi}{a^3} & 0<a \end{cases} \qquad (1)$$

$> A := \left(\frac{2\cdot a^3}{Pi}\right)^{\left(\frac{1}{2}\right)};$

$$A := \sqrt{2}\sqrt{\frac{a^3}{\pi}} \qquad (2)$$

$> assume(a>0);$

$> int\left(\frac{A^2\cdot x^2}{(x^2+a^2)^2}, x=-infinity..\infty\right);$

$$a\sim^2 \qquad (3)$$

$> -\frac{2\cdot A}{(2\cdot Pi\cdot h)^{\left(\frac{1}{2}\right)}}\cdot int\left(\frac{cos\left(\frac{p\cdot x}{h}\right)}{(x^2+a^2)}, x=0..infinity\right);$

$$\frac{1}{\sqrt{\pi h}\,a\sim}\left(\sqrt{\frac{a\sim^3}{\pi}}\left(\mathrm{I}\cosh\left(\frac{a\sim p}{h}\right)Ci\left(-\frac{\mathrm{I}\,a\sim p}{h}\right)-\mathrm{I}\cosh\left(\frac{a\sim p}{h}\right)Ci\left(\frac{\mathrm{I}\,a\sim p}{h}\right)\right.\right. \qquad (4)$$
$$\left.\left.-\operatorname{csgn}\left(\frac{p}{h}\right)\pi\sinh\left(\frac{a\sim p}{h}\right)\right)\right)$$

$> simplify(\%);$

$$-\sqrt{a\sim}\sqrt{\frac{1}{h}}\operatorname{csgn}\left(\frac{p}{h}\right)\left(-\cosh\left(\frac{a\sim p}{h}\right)+\sinh\left(\frac{a\sim p}{h}\right)\right) \qquad (5)$$

$> \frac{2\cdot a}{h}\cdot int\left(p^2\cdot exp\left(\frac{-2\cdot p\cdot a}{h}\right), p=0..infinity\right);$

$$\frac{2\,a\sim\left(\lim_{p\to\infty}\left(-\frac{1}{4}\,\frac{h\left(e^{\left(-\frac{2a\sim p}{h}\right)}h^2+2\,e^{\left(-\frac{2a\sim p}{h}\right)}p\,h\,a\sim+2\,e^{\left(-\frac{2a\sim p}{h}\right)}p^2a\sim^2-h^2\right)}{a\sim^3}\right)\right)}{h} \qquad (6)$$

$> simplify(\%);$

$$\frac{2\,a\sim\left(\lim_{p\to\infty}\left(-\frac{1}{4}\,\frac{h\left(e^{\left(-\frac{2a\sim p}{h}\right)}h^2+2\,e^{\left(-\frac{2a\sim p}{h}\right)}p\,h\,a\sim+2\,e^{\left(-\frac{2a\sim p}{h}\right)}p^2a\sim^2-h^2\right)}{a\sim^3}\right)\right)}{h} \qquad (7)$$

$>$

A1 Both are Hermitian because

$$(\hat{H}^T)^* = \hat{H}^\dagger = \hat{H}$$

and $$(\hat{A}^T)^* = \hat{A}^\dagger = \hat{A}$$

This implies the eigenvalues should be real and the eigenvectors should span, which they do.

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We will need the eigenvalues and eigenvectors of both matrices

<u>Eigenvalues $\hat{H}$</u> $0, \hbar\omega, 2\hbar\omega$

$$|0\hbar\omega\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} \qquad |\hbar\omega\rangle = \cancel{\frac{1}{\sqrt{2}}}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$|2\hbar\omega\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$$

<u>Eigenvalues $\hat{A}$</u> $0, \hbar, 2\hbar$

$$|0\hbar\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix} \qquad |\hbar\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|2\hbar\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix}$$

$\hat{A}$

eigenvectors {{1, 0, i}, {0, 1, 0}, {-i, 0, 1}}

Assuming i is the imaginary unit | Use i as a variable instead

Input:

Mathematica form

$$\text{Eigenvectors}\left[\begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 1 \end{pmatrix}\right]$$

$i$ is the imaginary unit »

Result:

Exact form

$$v_1 \approx \{i, 0, 1\}$$

$$v_2 \approx \{0, 1, 0\}$$

$$v_3 \approx \{-i, 0, 1\}$$

Corresponding eigenvalues:

$$\lambda_1 = 2$$

$$\lambda_2 = 1$$

$$\lambda_3 = 0$$

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$\hat{H}$

eigenvectors {{1, i, 0}, {-i, 1, 0}, {0, 0, 1}}

Assuming i is the imaginary unit | Use i as a variable instead

Input:

Mathematica form

$$\text{Eigenvectors}\left[\begin{pmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

$i$ is the imaginary unit »

Result:

Exact form

$v_1 \approx \{i, 1, 0\}$

$v_2 \approx \{0, 0, 1\}$

$v_3 \approx \{-i, 1, 0\}$

Corresponding eigenvalues:

$\lambda_1 = 2$

$\lambda_2 = 1$

$\lambda_3 = 0$

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In ground state

A2

$$\sigma_E^2 \sigma_a^2 \geq \left( \frac{1}{2i} \langle 0 | [\hat{H}, \hat{A}] | 0 \rangle \right)^2$$

Compute commutator

$$\hat{H}\hat{A} = \hbar^2\omega \begin{pmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 1 \end{pmatrix}$$

$$= \hbar^2\omega \begin{pmatrix} 1 & i & i \\ -i & 1 & 1 \\ -i & 0 & 1 \end{pmatrix}$$

$$\hat{A}\hat{H} = \hbar^2\omega \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \hbar^2\omega \begin{pmatrix} 1 & i & i \\ -i & 1 & 0 \\ -i & 1 & 1 \end{pmatrix}$$

$$[\hat{H}, \hat{A}] = \hat{H}\hat{A} - \hat{A}\hat{H} = \hbar^2\omega \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$\langle 0 | [\hat{H}, \hat{A}] | 0 \rangle$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 & 0 \end{pmatrix} \hbar^2 \omega \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}}$$

$$= \frac{\hbar^2 \omega}{2} \begin{pmatrix} i & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = 0$$

For ground state, $\sigma_E^2 \sigma_a^2 \geq 0$

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First excited state, $\langle \hbar\omega | [\hat{H}, \hat{A}] | \hbar\omega \rangle$

$$= \begin{pmatrix} 0, & 0 & 1 \end{pmatrix} \hbar^2 \omega \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \hbar^2 \omega \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$$

$\sigma_E^2 \sigma_a^2 \geq 0$

A3 The energy ground state from A1

$$|0\hbar\omega\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} = |\psi\rangle$$

The eigenvectors of $\hat{A}$ also from A1

$$|0\hbar\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix} \qquad |\hbar\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|2\hbar\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix}$$

The probability to observe $0\hbar$ is

$$P(0\hbar) = |\langle 0\hbar|\psi\rangle|^2$$

$$\langle 0\hbar|\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 0 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}\cdot 1 = \frac{1}{2}$$

$$P(0\hbar) = \langle 0\hbar|\psi\rangle^* \cdot \langle 0\hbar|\psi\rangle = \frac{1}{4}$$

$$\langle \hbar | \psi \rangle = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}$$

$$P(\hbar) = \langle \hbar | \psi \rangle^* \langle \hbar | \psi \rangle = \frac{1}{2}$$

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$$\langle 2\hbar | \psi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{2} \cdot (-1) = -\frac{1}{2}$$

$$P(2\hbar) = \langle 2\hbar | \psi \rangle^2 \langle 2\hbar | \psi \rangle = \frac{1}{4}$$

| Outcome (a) | Probability |
|---|---|
| $0\hbar = 0$ | $\frac{1}{4}$ |
| $\hbar$ | $\frac{1}{2}$ |
| $2\hbar$ | $\frac{1}{4}$ |
| Anything | 1 |

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$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\hbar\omega\rangle + \frac{1}{\sqrt{2}}|\hbar\omega\rangle$$

$$|\psi(t)\rangle = \hat{U}|\psi(0)\rangle$$

$$= \frac{1}{\sqrt{2}}|0\hbar\omega\rangle + \frac{1}{\sqrt{2}}e^{-i\omega t}|\hbar\omega\rangle$$

$$= \frac{1}{2}\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}e^{-i\omega t}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -i/2 \\ 1/2 \\ e^{-i\omega t}/\sqrt{2} \end{pmatrix}$$

Expectation Value

$$\langle o\rangle(t) = \langle\psi|\hat{A}|\psi\rangle$$

$$= \begin{pmatrix} \frac{i}{2} & \frac{1}{2} & \frac{e^{i\omega t}}{\sqrt{2}} \end{pmatrix} \hbar \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 1 \end{pmatrix} \begin{pmatrix} -i/2 \\ 1/2 \\ \frac{e^{-i\omega t}}{\sqrt{2}} \end{pmatrix}$$

$$\langle a\rangle(t) = \hbar\begin{pmatrix} \frac{i}{2} & \frac{1}{2} & \frac{e^{i\omega t}}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} -\frac{i}{2} + \frac{ie^{-i\omega t}}{\sqrt{2}} \\ \frac{1}{2} \\ -\frac{1}{2} + \frac{e^{-i\omega t}}{\sqrt{2}} \end{pmatrix}$$

$$= \hbar\left[\frac{1}{4} - \frac{e^{-i\omega t}}{2\sqrt{2}} + \frac{1}{4} - \frac{e^{i\omega t}}{2\sqrt{2}} + \frac{1}{2}\right]$$

$$= \hbar\left[1 - \frac{1}{2\sqrt{2}}\left(e^{i\omega t} + e^{-i\omega t}\right)\right]$$

$$= \hbar\left[1 - \frac{1}{\sqrt{2}}\cos\omega t\right]$$

A5

$$|\psi\rangle = |0\hbar\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix}$$

## Expectation Value of E

$$\langle E\rangle = \langle\psi|H|\psi\rangle$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} i & 0 & 1 \end{pmatrix} \hbar\omega \begin{pmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{\hbar\omega}{2}\begin{pmatrix} i & 0 & 1 \end{pmatrix}\begin{pmatrix} -i \\ -1 \\ 1 \end{pmatrix}$$

$$= \frac{\hbar\omega}{2}(1+1) = \hbar\omega$$