PHYS 4073 - Quantum Mechanics- Homework Set 7

Reading Assignment: No additional material.

Due at 5:45pm Monday November 1st in my box or at my office.

Griffiths' Problems

Work four of the seven following Griffiths' problems

2.10 - Use raising operator to construct ψ_2 from ψ_1.

2.13

2.14

2.15

2.16

2.41

25 pts each

2.10

$$\phi_0(x) = A e^{-\frac{m\omega}{2\hbar} x^2}$$

$$A = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

$$\hat{a}_+ = \frac{1}{\sqrt{2\hbar m\omega}} \left(-\hbar \frac{d}{dx} + m\omega x \right)$$

$$\hat{a}_+ |\phi_n\rangle = \sqrt{n+1}\, |\phi_{n+1}\rangle$$

$$|\phi_1\rangle = \hat{a}_+ |\phi_0\rangle$$

$$|\phi_2\rangle = \frac{\hat{a}_+}{\sqrt{2}} |\phi_1\rangle$$

Using Maxima

$$\langle x|\phi_1\rangle = \sqrt{\frac{2m\omega}{\hbar}} \cdot A \cdot x\, e^{-\frac{m\omega}{2\hbar} x^2} \quad ✓$$

$$\langle x|\phi_2\rangle = A\left(\sqrt{2}\, \frac{m\omega}{\hbar} x^2 - \frac{1}{\sqrt{2}} \right) e^{-\frac{m\omega}{2\hbar} x^2}$$

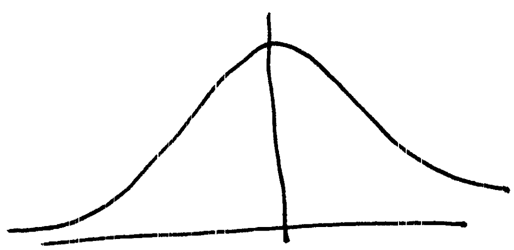
Or in terms of the dimensionless, $\xi = \sqrt{\frac{m\omega}{\hbar}}$

$$\phi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{2}\, \xi \, e^{-\xi^2/2}$$

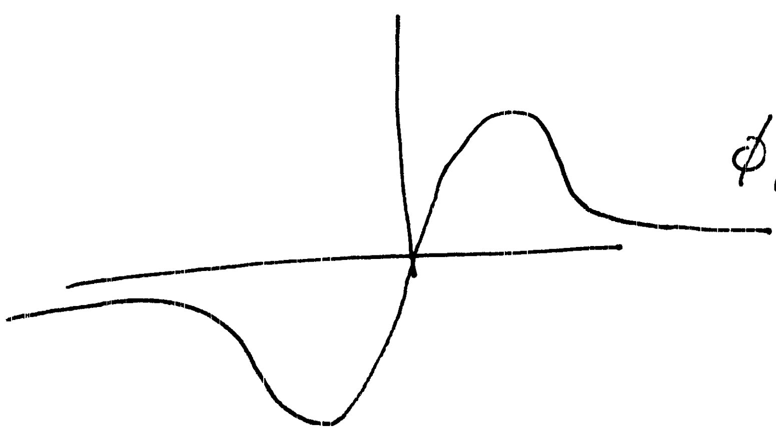
$$\phi_2(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \left(\sqrt{2}\, \xi^2 - \frac{1}{\sqrt{2}}\right) e^{-\xi^2/2}$$

which are correct.

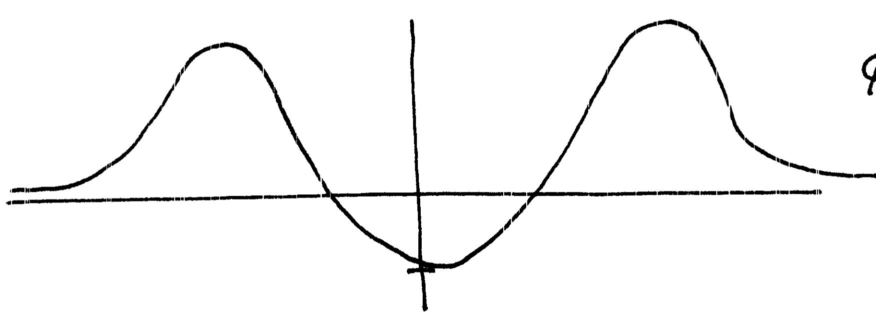
(b)



ϕ_0



ϕ_1



ϕ_2

(c) To check orthogonality, check

$$\int_{-\infty}^{\infty} dx \, \phi_i^* \phi_j = \delta_{ij}$$

which checks using maxima.

(%i75)
```
phi0: A*exp(-m*omega*x^2/(2*hbar));
```

(%o75) $$\%e^{-\frac{m\,\omega\,x^2}{2\,hbar}}\,A$$

(%i76)
```
aplus(u):=(-hbar*diff(u, x)+m*omega*x*u)/sqrt(2*hbar*m*omega);
```

(%o76) $$\mathrm{aplus}(u):=\frac{(-hbar)\,\mathrm{diff}(u,x)+m\,\omega\,x\,u}{\sqrt{2\,hbar\,m\,\omega}}$$

(%i77)
```
phi1: aplus(phi0);
```

(%o77) $$\frac{\sqrt{2}\,\sqrt{m}\,\sqrt{\omega}\,x\,\%e^{-\frac{m\,\omega\,x^2}{2\,hbar}}\,A}{\sqrt{hbar}}$$

(%i78)
```
phi2:aplus(phi1)/sqrt(2);
collectterms(%);
```

(%o78)
$$\frac{\frac{\sqrt{2}\,m^{3/2}\,\omega^{3/2}\,x^2\,\%e^{-\frac{m\,\omega\,x^2}{2\,hbar}}\,A}{\sqrt{hbar}}-hbar\left(\frac{\sqrt{2}\,\sqrt{m}\,\sqrt{\omega}\,\%e^{-\frac{m\,\omega\,x^2}{2\,hbar}}\,A}{\sqrt{hbar}}-\frac{\sqrt{2}\,m^{3/2}\,\omega^{3/2}\,x^2\,\%e^{-\frac{m\,\omega\,x^2}{2\,hbar}}\,A}{hbar^{3/2}}\right)}{2\,\sqrt{hbar}\,\sqrt{m}\,\sqrt{\omega}}$$

(%o79)
$$\frac{\frac{\sqrt{2}\,m^{3/2}\,\omega^{3/2}\,x^2\,\%e^{-\frac{m\,\omega\,x^2}{2\,hbar}}\,A}{\sqrt{hbar}}-hbar\left(\frac{\sqrt{2}\,\sqrt{m}\,\sqrt{\omega}\,\%e^{-\frac{m\,\omega\,x^2}{2\,hbar}}\,A}{\sqrt{hbar}}-\frac{\sqrt{2}\,m^{3/2}\,\omega^{3/2}\,x^2\,\%e^{-\frac{m\,\omega\,x^2}{2\,hbar}}\,A}{hbar^{3/2}}\right)}{2\,\sqrt{hbar}\,\sqrt{m}\,\sqrt{\omega}}$$

(%i80)
```
load(format);
```

(%o80)
C:/PROGRA~2/MAXIMA~1.1/share/maxima/5.22.1/share/contrib/format/format.mac

(%i81)
```
format(phi2, %poly(x));
```

(%o81) $$\frac{\sqrt{2}\,m\,\omega\,x^2\,\%e^{-\frac{m\,\omega\,x^2}{2\,hbar}}\,A}{hbar}-\frac{\%e^{-\frac{m\,\omega\,x^2}{2\,hbar}}\,A}{\sqrt{2}}$$

(%i82)
```
integrate(phi0*phi1, x, minf, inf);
```

(%o82) 0

(%i83)
```
integrate(phi0*phi2, x, minf, inf);
```

(%o83) 0

```
(%i84) integrate(phi2*phi1, x, minf, inf);
(%o84) 0
```

(2.13) Assume we use normalized ψ_0, ψ_1, then

(a) $$1 = \sqrt{(3A)^2 + (4A)^2} = A\sqrt{25} = 5A$$

$$A = \frac{1}{5}$$

$$\psi(x,0) = \frac{3}{5}\psi_0 + \frac{4}{5}\psi_1$$

(b) $$E_0 = \frac{\hbar\omega}{2} \qquad E_1 = \frac{3\hbar\omega}{2}$$

$$\psi(x,t) = \frac{3}{5}e^{-\frac{iE_0 t}{\hbar}}\psi_0 + \frac{4}{5}e^{-\frac{iE_1 t}{\hbar}}\psi_1$$

$$= \frac{3}{5}e^{-\frac{i\omega t}{2}}\psi_0 + \frac{4}{5}e^{-\frac{3i\omega t}{2}}\psi_1$$

$$|\psi(x,t)|^2 = \psi^*\psi$$

$$= \frac{1}{25}\left(3e^{i\omega t/2}\psi_0 + 4e^{3i\omega t/2}\psi_1\right)\cdot$$

$$\left(3e^{-i\omega t/2}\psi_0 + 4e^{-3i\omega t/2}\psi\right)$$

$$|\psi(x,t)|^2 = \frac{1}{25}\left(9\psi_0 + 16\psi_1^2 + 12\psi_0\psi_1\left(e^{i\omega t} + e^{-i\omega t}\right)\right)$$

$$= \frac{9}{25}\psi_0 + \frac{16}{25}\psi_1^2 + \frac{24}{25}\psi_0\psi_1 \cos\omega t$$

(c)

$$\langle x \rangle = \langle \psi | \hat{x} | \psi \rangle$$

$$= \int_{-\infty}^{\infty} x\, |\psi(x,t)|^2\, dx$$

$$= \frac{9}{25}\int x\, \psi_0^2\, dx$$

$$+ \frac{16}{25}\int x\, \psi_1^2\, dx$$

$$+ \frac{24}{25}\cos\omega t \int dx\, x\, \psi_0\psi_1$$

The first two terms are the average location in the stationary state.

$$\langle \phi_0 | \hat{x} | \phi_0 \rangle = \langle \phi_1 | \hat{x} | \phi_1 \rangle = 0$$

Using functions found in 2.10.

$$\int dx\, x\, \psi_0 \psi_1 = A^2 \sqrt{\frac{\pi}{2}} \frac{\hbar}{m\omega} \qquad \text{(maxima)}$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \left(\frac{\pi}{2}\right)^{1/2} \frac{\hbar}{m\omega}$$

$$= \sqrt{\frac{\hbar}{2m\omega}}$$

$$\langle x \rangle (t) = \frac{24}{25} \sqrt{\frac{\hbar}{2m\omega}} \cos\omega t$$

Since we have $\langle x \rangle$ as a function of time

$$m \frac{d\langle x \rangle}{dt} = \langle p \rangle = -\frac{24}{25} m\omega \sqrt{\frac{\hbar}{2m\omega}} \sin\omega t$$

$$= -\frac{24}{25} \sqrt{\frac{\hbar m\omega}{2}} \sin\omega t$$

<u>Ehrenfest's Thm</u>

$$\frac{d\langle p \rangle}{dt} = \left\langle -\frac{\partial V}{\partial x} \right\rangle \qquad V = \frac{1}{2} m\omega^2 x$$

$$\frac{\partial V}{\partial x} = m\omega^2 x$$

$$\left\langle -\frac{\partial V}{\partial x} \right\rangle = -m\omega^2 \langle x \rangle = -\frac{24}{25} m\omega^2 \sqrt{\frac{\hbar}{2m\omega}} \cos\omega t$$

and

$$\frac{d\langle p\rangle}{dt} = -\frac{24}{25}\, m\omega^2 \sqrt{\frac{\hbar}{2m\omega}}\, \cos\omega t$$ ✓

(d) $P\left(\frac{\hbar\omega}{2}\right) = c_0^* c_0 = \frac{9}{25}$

$P\left(\frac{3\hbar\omega}{2}\right) = c_1^* c_1 = \frac{16}{25}$

(2.14) $|\psi\rangle = |\frac{\hbar\omega}{2}\rangle$

New energy spectrum

$$E_n' = \left(n+\frac{1}{2}\right)\hbar\omega' = 2\left(n+\frac{1}{2}\right)\hbar\omega$$

The lowest possible energy with the new spring constant is $\hbar\omega$, so $\frac{\hbar\omega}{2}$ cannot be observed.

$$\langle x|\psi\rangle = \underbrace{\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}}_{A} e^{-\frac{m\omega x^2}{2\hbar}}$$ state function

New ground state, ϕ_0'

$$\phi_0' = \left(\frac{2m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{\hbar}}$$

$$c_0' = \int_{-\infty}^{\infty} dx\, \phi_0'^{*} \phi_0$$

$$= 2^{1/4} A^2 \int_{-\infty}^{\infty} e^{-\frac{m\omega}{2\hbar}x^2} e^{-\frac{m\omega}{\hbar}x^2}\, dx$$

$$C_0' = 2^{3/4} \sqrt{\frac{\pi \hbar}{3 m \omega}} \cdot \left(\frac{m \omega}{\pi \hbar} \right)^{1/2}$$

$$= 2^{3/4} \sqrt{\frac{1}{3}}$$

$$P(\hbar \omega) = C_0' C_0 = \frac{2^{3/2}}{3} = 0.94$$

(%i75) phi0: A*exp(-m*omega*x^2/(2*hbar));

(%o75) $\%e^{-\frac{m\,\omega\,x^2}{2\,hbar}}\,A$

(%i76) aplus(u):=(-hbar*diff(u, x)+m*omega*x*u)/sqrt(2*hbar*m*omega);

(%o76) $aplus(u):=\frac{(-hbar)\,\mathrm{diff}(u,x)+m\,\omega\,x\,u}{\sqrt{2\,hbar\,m\,\omega}}$

(%i77) phi1: aplus(phi0);

(%o77) $\frac{\sqrt{2}\,\sqrt{m}\,\sqrt{\omega}\,x\,\%e^{-\frac{m\,\omega\,x^2}{2\,hbar}}\,A}{\sqrt{hbar}}$

(%i78) phi2:aplus(phi1)/sqrt(2);
collectterms(%);

(%o78)

$$\frac{\frac{\sqrt{2}\,m^{3/2}\,\omega^{3/2}\,x^2\,\%e^{-\frac{m\,\omega\,x^2}{2\,hbar}}\,A}{\sqrt{hbar}}-hbar\left(\frac{\sqrt{2}\,\sqrt{m}\,\sqrt{\omega}\,\%e^{-\frac{m\,\omega\,x^2}{2\,hbar}}\,A}{\sqrt{hbar}}-\frac{\sqrt{2}\,m^{3/2}\,\omega^{3/2}\,x^2\,\%e^{-\frac{m\,\omega\,x^2}{2\,hbar}}\,A}{hbar^{3/2}}\right)}{2\,\sqrt{hbar}\,\sqrt{m}\,\sqrt{\omega}}$$

(%o79)

$$\frac{\frac{\sqrt{2}\,m^{3/2}\,\omega^{3/2}\,x^2\,\%e^{-\frac{m\,\omega\,x^2}{2\,hbar}}\,A}{\sqrt{hbar}}-hbar\left(\frac{\sqrt{2}\,\sqrt{m}\,\sqrt{\omega}\,\%e^{-\frac{m\,\omega\,x^2}{2\,hbar}}\,A}{\sqrt{hbar}}-\frac{\sqrt{2}\,m^{3/2}\,\omega^{3/2}\,x^2\,\%e^{-\frac{m\,\omega\,x^2}{2\,hbar}}\,A}{hbar^{3/2}}\right)}{2\,\sqrt{hbar}\,\sqrt{m}\,\sqrt{\omega}}$$

(%i80) load(format);

(%o80)
C:/PROGRA~2/MAXIMA~1.1/share/maxima/5.22.1/share/contrib/format/format.mac

(%i81) format(phi2, %poly(x));

(%o81) $\frac{\sqrt{2}\,m\,\omega\,x^2\,\%e^{-\frac{m\,\omega\,x^2}{2\,hbar}}\,A}{hbar}-\frac{\%e^{-\frac{m\,\omega\,x^2}{2\,hbar}}\,A}{\sqrt{2}}$

(%i82) integrate(phi0*phi1, x, minf, inf);

(%o82) 0

(%i83) integrate(phi0*phi2, x, minf, inf);

(%o83) 0

(%i84) integrate(phi2*phi1, x, minf, inf);

(%o84) 0

(%i85) integrate(x*phi0*phi1, x, minf, inf);

(%o85) $\frac{\sqrt{\pi}\, hbar\, A^2}{\sqrt{2\, m\, \omega}}$

(%i86) integrate(x*phi0*phi0, x, minf, inf);

(%o86) 0

(%i87) integrate(x*phi1*phi1, x, minf, inf);

(%o87) 0

(%i88) phi0p: 2^(1/4)*A*exp(-m*omega*x^2/(hbar));

(%o88) $2^{1/4}\, \%e^{-\frac{m\, \omega\, x^2}{hbar}}\, A$

(%i89) integrate(phi0*phi0p, x, minf, inf);

(%o89) $\frac{2^{3/4}\, \sqrt{\pi}\, \sqrt{hbar}\, A^2}{\sqrt{3}\, \sqrt{m}\, \sqrt{\omega}}$

(2.15) Define $x_c = \sqrt{\frac{2E}{m\omega^2}}$

the classical turning point

In the ground state, $E = \frac{1}{2}\hbar\omega$

$$x_c = \sqrt{\frac{\hbar}{m\omega}}$$

The normalized ground state wave function is

$$\phi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}}$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-x^2/2x_c^2}$$

$$= \frac{1}{\pi^{1/4}} \frac{1}{\sqrt{x_c}} e^{-x^2/2x_c^2}$$

Calculate $\mathcal{P}(x \in [-x_c, x_c])$

$$= \frac{1}{\sqrt{\pi}} \int_{-x_c}^{x_c} \frac{dx}{x_c} e^{-x^2/x_c^2} dx \qquad = \int_{-x_c}^{x_c} \phi_0^* \phi_0 \, dx$$

Let $u = x/x_c$

$$\mathcal{P}(x \in [-x_c, x_c]) = \frac{2}{\sqrt{\pi}} \int_0^1 e^{-u^2} du$$

$$= \text{erf}(1) = 0.84270$$

$$\mathcal{P}(x \text{ outside classical range}) = 1 - \text{erf}(1)$$

$$= 0.1573$$

> $int\left(exp\left(-x^2\right),\ x=0..1\right)$

$$\frac{1}{2}\,\mathrm{erf}(1)\sqrt{\pi} \qquad (1)$$

> $int\left(exp\left(-x^2\right),\ x=0..1\right)\cdot\frac{2}{sqrt(Pi)}$

$$\mathrm{erf}(1) \qquad (2)$$

> $evalf(\%)$:

$$0.8427007929 \qquad (3)$$

> $1-\%$

$$0.1572992071 \qquad (4)$$

>

(2.16) Recursion Formula

$$a_{j+2} = \frac{-2(n-j)}{(j+1)(j+2)} a_j$$

H_5 : $a_0 = 0$, $a_1 = 1$, $n = 5$

$$a_3 = \frac{-2(5-1)}{(1+1)(1+2)} \cdot 1 = -\frac{4}{3}$$

$$a_5 = \frac{-2(5-3)}{(3+1)(3+2)} \left(-\frac{4}{3}\right) = \frac{-4}{4 \cdot 5} \cdot \left(-\frac{4}{3}\right) = \frac{4}{15}$$

$$H_5(x) = \frac{4}{15}x^5 - \frac{4}{3}x^3 + x$$

Coefficient of x^5 is $2^5 = 32$. Multiply by $\frac{15}{4} \cdot 32 = 15 \cdot 8$

$$H_5(x) = 32x^5 - 160x^3 + 120x$$

H_6 : $a_0 = 1$ $a_1 = 0$ $n = 6$

$$a_2 = \frac{-2(6-0)}{(0+1)(0+2)} 1 = -6$$

$$a_4 = \frac{-2(6-2)}{(2+1)(2+2)} (-6) = \frac{12 \cdot 4}{3 \cdot 4} = 4$$

$$a_6 = \frac{-2(6-4)}{(4+1)(4+2)} \cdot 4 = \frac{-16}{5 \cdot 6}$$

$$= -\frac{8}{15}$$

Make coefficient of x^6 be $2^6 = 64$. Multiply by $-\frac{15}{8} \cdot 64 = -15 \cdot 8$

$$64x^6 - 480x^4 + 720x^2 - 120 = 0$$

(2.41)

$$\phi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\xi^2/2} = B e^{-\xi^2/2}$$

$$\phi_1 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{2\xi}{\sqrt{2}} e^{-\xi^2/2} = B\sqrt{2}\,\xi\, e^{-\xi^2/2}$$

$$\phi_2 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{4\xi^2 - 2}{2\sqrt{2}} e^{-\xi^2/2}$$

from 2.85

$$= B\left(\sqrt{2}\,\xi^2 - \frac{1}{\sqrt{2}}\right) e^{-\xi^2/2}$$

$$\xi = \sqrt{\frac{m\omega}{\hbar}}\, x \qquad B \equiv \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

The given wave function is

$$\psi(x,0) = A(1-2\xi)^2 e^{-\xi^2/2}$$

$$= A(1 - 4\xi + 4\xi^2)\, e^{-\xi^2/2}$$

$$= c_0\phi_0 + c_1\phi_1 + c_2\phi_2$$

$$\Rightarrow 4A = c_2 B\sqrt{2}$$

Equate powers of ξ

$$\boxed{c_2 = 2\sqrt{2}\,\frac{A}{B}}$$

$$-4A = B\sqrt{2}\, c_1$$

$$\boxed{c_1 = -2\sqrt{2}\,\frac{A}{B}}$$

ξ^0:

$$A = c_0 B - c_2 \frac{B}{\sqrt{2}}$$

$$= c_0 B - 2\sqrt{2}\,\frac{A}{B}\left(\frac{B}{\sqrt{2}}\right)$$

$$= c_0 B - 2A$$

$$\boxed{c_0 = 3\frac{A}{B}}$$

<u>Normalize to find A</u>

$$c_0^* c_0 + c_1^* c_1 + c_2^* c_2 = 1$$

$$\frac{9A^2}{B^2} + \frac{8A^2}{B^2} + \frac{8A^2}{B^2} = 1$$

$$A = \frac{B}{\sqrt{25}} = \frac{B}{5}$$

$c_0 = \frac{3}{5} \qquad c_1 = -\frac{2\sqrt{2}}{5} \qquad c_2 = \frac{2\sqrt{2}}{5}$

$$\psi = \frac{3}{5}\phi_0 - \frac{2\sqrt{2}}{5}\phi_1 + \frac{2\sqrt{2}}{5}\phi_2$$

$$\langle E \rangle = c_0^* c_0 \left(\frac{1}{2}\hbar\omega\right) + c_1^* c_1 \left(\frac{3}{2}\hbar\omega\right) + c_2^* c_2 \left(\frac{5}{2}\hbar\omega\right)$$

$$= \frac{9}{25}\left(\frac{1}{2}\hbar\omega\right) + \frac{8}{25}\left(\frac{3}{2}\hbar\omega\right) + \frac{8}{25}\left(\frac{5}{2}\hbar\omega\right)$$

$$= \frac{\hbar\omega}{50}\left(9 + 24 + 40\right)$$

$$\boxed{\langle E \rangle = \frac{73}{50}\hbar\omega}$$

(b) $\psi(x,t) = \frac{3}{5} e^{-i\omega t/2}\phi_0 - \frac{2\sqrt{2}}{5} e^{-3i\omega t/2}\phi_1 + \frac{2\sqrt{2}}{5} e^{-5i\omega t/2}\phi_2$

~~The change affects only the ϕ_1 term changing its sign~~

~~$\frac{3\omega T}{2}$~~

Factor out the overall phase

$$\Psi(x,t) = e^{-i\omega t/2}\left[\frac{3}{5}\phi_0 - \frac{2\sqrt{2}}{5}e^{-i\omega t}\phi_1 + \frac{2\sqrt{2}}{5}e^{-2i\omega t}\phi_2\right]$$

We want to change the sign of the second term without affecting the third term

$$\omega T = \pi$$

$$T = \frac{\pi}{\omega}$$