PHYS 4073 - Quantum Mechanics- Homework Set 8

Reading Assignment: Chapter 4 Section 1 and 2

Due at 5:45pm Monday November 8th in my box or at my office.

Griffiths' Problems

4.1 part (a)

4.2 part (a) and (b)

4.3

4.8

4.11

4.13

4.15

4.38 part (a)

4.1

$$\hat{X}_1 = x \qquad \hat{X}_2 = y \qquad \hat{X}_3 = z$$

$$[\hat{X}_i, \hat{X}_j] f = (xy - yx) f = 0$$

for all combinations x, y, z

$$\hat{P}_1 = \frac{\hbar}{i}\frac{\partial}{\partial x} \qquad \hat{P}_2 = \frac{\hbar}{i}\frac{\partial}{\partial y} \qquad \hat{P}_3 = \frac{\hbar}{i}\frac{\partial}{\partial z}$$

$$[\hat{P}_1, \hat{P}_2] f = -\hbar^2 \left[\frac{\partial^2}{\partial x \partial y} - \frac{\partial^2}{\partial y \partial x}\right] f = 0$$

for all combinations of x, y.

$[\hat{X}_i, \hat{P}_j] = i\hbar$ if $i = j$. We have already proved this part

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~~If $i \neq j$~~

$$\cancel{[\hat{X}_1, \hat{P}_2] f = \left(x \frac{\hbar}{i}\frac{\partial}{\partial y} - y \frac{\hbar}{i}\frac{\partial}{\partial x}\right) f}$$

If $i \neq j$

$$[\hat{X}_1, \hat{P}_2] = \left(x \frac{\hbar}{i} \frac{\partial}{\partial y} - \frac{\hbar}{i} \frac{\partial}{\partial y} x \right) f$$

$$= 0$$

For all combinations of x, y, z s.t. $i \neq j$

(b) Ehrenfest's Thm did not depend on the number of dimensions.

$$\frac{d}{dt} \langle \vec{r} \rangle = \frac{i}{\hbar} \langle [\hat{H}, \vec{R}] \rangle$$

Break this down into components. Do the x-component

$$\frac{d}{dt} \langle \hat{X} \rangle = \frac{i}{\hbar} \langle [\hat{H}, \hat{X}] \rangle$$

$$= \frac{i}{\hbar} \frac{1}{2m} \langle [\hat{P}_x^2, \hat{X}] \rangle$$

$$= \frac{1}{m} \langle P_x \rangle$$

Since $V(x, y, z)$ commutes with $\hat{X}$ as does P_y^2, P_z^2

Likewise,

$$\frac{d}{dt}\langle P_x\rangle = \frac{i}{\hbar}\langle [H, P_x]\rangle$$

$$= \frac{i}{\hbar}\langle [V(x,y,z), P_x]\rangle$$

Commuting the P_x will differentiate with respect to x while leaving y, z alone, thus taking a partial derivative

$$\frac{d}{dt}\langle P_x\rangle = -\left\langle \frac{\partial V}{\partial x}\right\rangle$$

(c) Since $[X_i, P_i] = i\hbar$

$$\sigma_x \sigma_{px} \geq \frac{\hbar}{2} \qquad \sigma_y \sigma_{py} \geq \frac{\hbar}{2} \qquad \sigma_z \sigma_{pz} \geq \hbar/2$$

and since $[X_i, P_j] = 0 \quad i \neq j$

all other uncertainties have the form

$$\sigma_x \sigma_{py} \geq 0$$

4.2 I did part of this in lecture

SE $H\phi = E\phi$

$$-\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\phi = E\phi$$

Let $\phi = \phi_x(x)\,\phi_y(y)\,\phi_z(z)$

$$\underbrace{-\frac{\hbar^2}{2m\phi_x}\frac{\partial^2\phi_x}{\partial x^2}}_{E_1} \underbrace{-\frac{\hbar^2}{2m\phi_y}\frac{\partial^2\phi_y}{\partial y^2}}_{E_2} \underbrace{-\frac{\hbar^2}{2m\phi_z}\frac{\partial^2\phi_z}{\partial z^2}}_{E_3} = E$$

$$E = E_1 + E_2 + E_3$$

$$\phi = \left(\sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}\right)\left(\sqrt{\frac{2}{a}}\sin\frac{m\pi x}{a}\right)\left(\sqrt{\frac{2}{a}}\sin\frac{l\pi x}{a}\right)$$

$$E_{nml} = \frac{\hbar^2}{2m}\left(\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{a}\right)^2 + \left(\frac{l\pi}{a}\right)^2\right)$$

$$= \frac{\hbar^2\pi^2}{2ma^2}\left(n^2 + m^2 + l^2\right)$$

(b) Let $E_0 \equiv \dfrac{\hbar^2 \pi^2}{2ma^2}$

Level	n, m, l	E	Degeneracy
1	1, 1, 1	$3E_0$	1
2	1, 1, 2	$6E_0$	3
3	1, 2, 2	$9E_0$	3
4	1, 1, 3	$11E_0$	3
5	2, 2, 2	$12E_0$	1
6	1, 2, 3	$14E_0$	6

(c) The state E_{14} is special because it can be formed by both

$3^2+3^2+3^2 = 27$ and $1^2+1^2+5^2 = 27$

This part was not required.

4.3

Construct Y_0^0

Eq 4.28

$$P_0(x) = \frac{1}{2^0 0!}\left(\frac{d}{dx}\right)^0 (x^2-1)^0 = 1$$

Eq 4.27

$$P_0^0(x) = (1-x^2)^{|0|/2}\left(\frac{d}{dx}\right)^{|0|} P_0$$

$$= 1$$

$$Y_0^0 = \varepsilon \sqrt{\frac{(2\cdot 0+1)(0-|0|)!}{4\pi\,(0+|0|)!}}\, e^{i0\phi} P_0^0$$

$$\varepsilon = (-1)^0 \qquad m \geq 0$$

$$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2}$$

Check normalized

$$1 = \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \,\sin\theta\, Y_0^{0*} Y_0^0 = \frac{4\pi}{4\pi} = 1$$

Construct Y_2^1

$$P_2 = \frac{1}{2^2\, 2!} \left(\frac{d}{dx}\right)^2 (x^2-1)^2$$

$$= \frac{1}{8} \frac{d^2}{dx^2} x^4 - 2x^2 + 1$$

$$= \frac{1}{8}\left(12x^2 - 4\right) = \frac{1}{2}\left(3x^2-1\right)$$

$$P_2^1 = \left(1-x^2\right)^{\frac{|1|}{2}} \frac{d}{dx} P_2(0)$$

$$= \sqrt{1-x^2} \cdot 3x$$

$$P_2^1(\cos\theta) = \sqrt{1-\cos^2\theta} \cdot 3\cos\theta$$

$$= 3\sin\theta\cos\theta = \frac{3}{2}\sin 2\theta$$

$$Y_2^1 = \varepsilon \sqrt{\frac{(2\cdot 2+1)}{4\pi}\,\frac{(2-1)!}{(2+1)!}}\; e^{i\phi}\, P_2^1(\cos\theta)$$

$$\varepsilon = (-1)^m = -1 \qquad m \geq 0$$

$$= -\sqrt{\frac{5}{4\pi\cdot 6}}\; e^{i\phi}\cdot 3\,\sin\theta\cos\theta$$

$$= -\sqrt{\frac{15}{8\pi}}\,\sin\theta\cos\theta\, e^{i\phi}$$

Check normalization

$$I = \int_0^{\pi} d\theta \int_0^{2\pi} d\phi\; Y_2^{1*}\, Y_2^1 \sin\theta$$

$$= \left(\frac{15}{8\pi}\right) \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \sin^3\theta \cos^2\theta\, d\theta$$

Do the ϕ integral to get 2π ~~and let $u = \cos\theta$~~ ~~$du = -\sin\theta\, d\theta$~~

~~$u = \sin\theta \quad du = \cos\theta\, d\theta$~~

$$\cancel{I = 2\pi\left(\frac{15}{8\pi}\right)\int_0^1 du}$$

$$\int_0^{\pi} d\theta \, \sin^3\theta \cos^2\theta = \frac{4}{15}$$ Alpha

$$1 = 2\pi\left(\frac{15}{8\pi}\right) \cdot \frac{4}{15} \checkmark$$

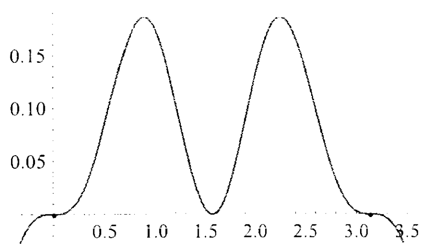
Wolfram Alpha

integrate sin^3(x)+cos^2(x) from x-0 to Pi

Definite integral:

$$\int_0^{\pi} \sin^3(x)\cos^2(x)\,dx = \frac{4}{15} \approx 0.266667$$

Visual representation of the integral:



4.8 Radial eqn $V=0,\ l=1$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2}+\frac{2\hbar^2}{2mr^2}u=Eu$$

$$\boxed{E=\frac{-k^2\hbar^2}{2m}}$$

$$\frac{d^2u}{dr^2}-\frac{2u}{r^2}=-\frac{2mE}{\hbar^2}u=-k^2u$$

$u = A r j_1(kr) =$ Use Maple

$$\frac{d^2u}{dr^2}-\frac{2u}{r^2}=-k^2\left(\frac{\sin kr}{k^2r}-\frac{\cos kr}{k}\right)A$$

$u = A r j_1(kr)$

$$= A\left(\frac{\sin kr}{k^2r}-\frac{\cos kr}{k}\right)$$

✓

(b) Allowed energies

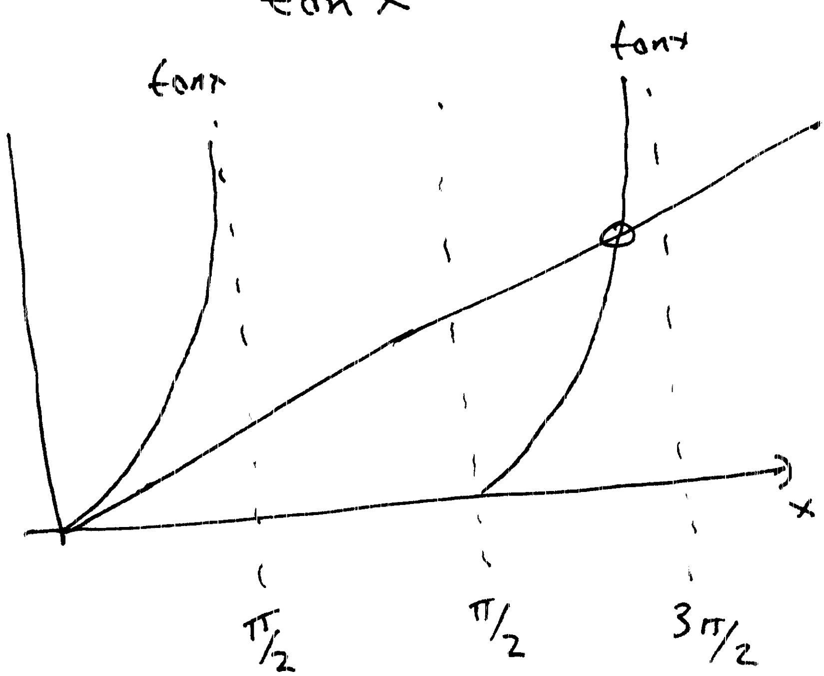
$$j_1(ka) = 0$$

$$j_1(kr) = \frac{\sin kr}{(kr)^2} - \frac{\cos kr}{kr}$$

$$\frac{\sin ka}{(ka)^2} - \frac{\cos ka}{ka} = 0$$

$$\frac{\sin ka}{\cos ka} = ka \qquad \text{Let } ka = x$$

$$\tan x = x$$



For large n, the two curves intersect at about $\left(n+\frac{1}{2}\right)\pi = x = ka$

$$k = \left(n+\frac{1}{2}\right)\frac{\pi}{a} \simeq \frac{n\pi}{a} \qquad n \text{ large}$$

$$E = \frac{\hbar^2\pi^2}{2m} = \frac{\hbar^2\pi^2 n}{2ma^2}$$

4.11

$$R_{20} = A\left(1 - \frac{r}{2a}\right) e^{-r/2a}$$

Normalize

$$1 = \int_0^\infty dr\, r^2 R^* R$$

$$= A^2 \int dr\, r^2 \left(1 - \frac{r}{2a}\right)^2 e^{-r/a}$$

Alpha

$$= A^2 \cdot 2a^3$$

$$A = \cancel{\sqrt{2}}\, \frac{1}{\sqrt{\cancel{8}}\, a^{3/2}}$$

$$R_{20} = \frac{1}{\sqrt{2}\, a^{3/2}} \left(1 - \frac{r}{2a}\right) e^{-r/2a}$$

Wolfram Alpha

integrate r^2*(1−r/(2*a))^2*e^(−r/a) from r =0 to infinity

Definite integral:

$$\int_0^\infty r^2 \left(1 - \frac{r}{2\,a}\right)^2 e^{-\frac{r}{a}}\, d\,r = 2\,a^3 \quad \text{if } \operatorname{Re}(a) > 0$$

(b) $R_{21} = A\, r e^{-r/2a}$

<u>Normalize</u>

$$1 = \int_0^\infty dr\, r^2 R^* R$$

$$= A^2 \int_0^\infty dr\, r^4 e^{-r/a}$$

$$= A^2 \cdot 24 a^5$$

$$A = \frac{1}{\sqrt{24 a^5}}$$ <u>Alpha</u>

$$R_{21} = \frac{1}{\sqrt{24 a^5}}\, r e^{-r/2a}$$

$$\psi_{211} = R_{21} Y_1^1$$

$$= \frac{-1}{\sqrt{24 a^3}} \frac{r}{a} e^{-r/2a} \left(\frac{3}{8\pi}\right)^{1/2} \sin\theta\, e^{i\phi}$$

$$\psi_{210} = R_{21} Y_1^0$$

$$= -\frac{1}{\sqrt{24a^3}} \frac{r}{a} e^{-r/2a} \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta$$

$$\psi_{21-1} = R_{21} Y_1^{-1}$$

$$= \frac{1}{\sqrt{24a^3}} \frac{r}{a} e^{-r/2a} \left(\frac{3}{8\pi}\right)^{1/2} \sin\theta e^{-i\phi}$$

Wolfram Alpha

integrate r^4+e^(−r/a) from r =0 to infinity

Definite integral

$$\int_0^\infty r^4\, e^{-\frac{r}{a}}\, dr = 24\, a^5$$

4.13

Ground state

$$\psi_{100} = R_{10} Y_0^0 = 2a^{-3/2} e^{-r/a} \sqrt{\frac{1}{4\pi}}$$

$$= A e^{-r/a} \qquad A = \frac{1}{\sqrt{\pi a^3}}$$

$$\langle r \rangle = \cancel{\int_0^\infty r\, r^2 dr \Big]_0}$$

$$= \int_0^\infty r\, r^2 dr\, R_{10}^* R_{10} \underbrace{\int_0^{2\pi} d\phi \int_0^\pi \sin\theta\, d\theta\, Y_0^{0*} Y_0^0}_{1}$$

$$\langle r \rangle = (A')^2 \int_0^\infty r^3 e^{-2r/a} dr$$

$$A' = 2a^{-3/2}$$

$$\langle r \rangle = \frac{3}{2} a$$

$$\langle r^2 \rangle = 3a^2 = (A')^2 \int_0^\infty r^4 e^{-2r/a} dr$$

4.13 b $\langle x \rangle = 0$ by symmetry

$\langle x^2 \rangle = \langle y^2 \rangle = \langle z^2 \rangle$ by symmetry

$\langle r^2 \rangle = \langle x^2 \rangle + \langle y^2 \rangle + \langle z^2 \rangle = 3\langle x^2 \rangle$

$$\langle x^2 \rangle = \frac{\langle r^2 \rangle}{3} = a^2$$

(c) $x^2 = r^2 \sin^2\theta \cos^2\phi$

$$\langle x^2 \rangle = \frac{1}{\pi a^3} \int_0^\infty dr\, r^4 e^{-2r/a} \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta \left(\sin^2\theta \cos^2\phi\right)$$

$$\int_0^\infty r^4 e^{-2r/a}\, dr =$$

$$\int_0^\pi d\theta$$

(c) $\psi_{211} = R_{21} Y_1^1$

$$R_{21} = \frac{1}{\sqrt{24}} a^{-3/2} \frac{r}{a} \exp\left(-\frac{r}{2a}\right)$$

$$Y_1^1 = -\left(\frac{3}{8\pi}\right)^{\frac{1}{2}} \sin\theta \, e^{i\phi}$$

$$\psi_{211} = -\left(\frac{1}{64\pi a^3}\right)^{1/2} \frac{r}{a} \exp\left(\frac{-r}{2a}\right) \sin\theta \, e^{i\phi}$$

$$\psi^*\psi = \frac{1}{64\pi a^5} r^2 \exp\left(-\frac{r}{a}\right) \sin^2\theta$$

$$x^2 = r^2 \sin^2\theta \cos^2\phi$$

$$\langle x^2 \rangle = \int x^2 \psi^* \psi \, dv$$

$$= \int_0^\infty dr \int_0^{2\pi} r \sin\theta \, d\phi \int_0^\pi r \, d\theta \; r^2 \sin^2\theta \cos^2\phi$$

$$\frac{1}{64\pi a^5} r^2 \exp\left(\frac{-r}{a}\right) \sin^2\theta$$

$$\langle x^2 \rangle = A \underbrace{\int_0^\infty r^6 e^{-r/a} dr}_{} \underbrace{\int_0^{2\pi} d\phi \cos^2\phi}_{} \underbrace{\int_0^\pi d\theta \sin^5\theta}_{16/15}$$

$$= A \cdot 720\, a^7 \qquad \pi$$

$$A = \frac{1}{64\pi a^5}$$

$$\langle x^2 \rangle = \frac{1}{64\pi a^5} \cdot 720\, a^7 \cdot \pi \cdot \frac{16}{15} = 12 a^2$$

(4.15)

$$\psi(r,0) = \frac{1}{\sqrt{2}} R_{21} \left(Y_1^1 + Y_1^{-1} \right)$$

$$Y_1^1 = -\left(\frac{3}{8\pi}\right)^{1/2} \sin\theta \, e^{i\phi}$$

$$Y_1^{-1} = \left(\frac{3}{8\pi}\right)^{1/2} \sin\theta \, e^{-i\phi}$$

$$Y_1^1 + Y_1^{-1} = \left(\frac{3}{8\pi}\right)^{1/2} \sin\theta \left(e^{-i\phi} - e^{i\phi} \right)$$

$$e^{i\phi} - e^{-i\phi} = 2i\sin\phi$$

$$Y_1^1 + Y_1^{-1} = \left(\frac{3}{8\pi}\right)^{1/2} \sin\theta \left(-2i\sin\phi \right)$$

$$= -\left(\frac{3}{2\pi}\right)^{1/2} i \sin\theta \sin\phi$$

$$\psi(r,0) = -i \left(\frac{3}{2\pi}\right)^{1/2} \frac{\sin\theta \sin\phi}{\sqrt{2}} \cdot \frac{1}{\sqrt{24}} a^{-3/2} \frac{r}{a} \exp\left(\frac{-r}{2a}\right)$$

$$= -i \left(\frac{1}{32\pi a^3}\right)^{1/2} \frac{r}{a} \exp\left(\frac{-r}{2a}\right) \sin\theta \sin\phi$$

The energy depends only on $n = 2$

$$E_{\blacksquare} = -\frac{\kappa^2 \hbar^2}{2m} \qquad \kappa = \frac{1}{an}$$

$$= -\frac{\hbar^2}{2a^2 m} \frac{1}{n^2}$$

$$E_2 = -\frac{\hbar^2}{8a^2 m}$$

$$\psi(r,t) = \psi(r,0)\, e^{-iE_2 t/\hbar}$$

(b) $$V(r) = -\frac{e^2}{4\pi\varepsilon_0 r}$$

$$\langle V \rangle = \frac{-e^2}{4\pi\varepsilon_0} \left\langle \frac{1}{r} \right\rangle$$

$$\left\langle \frac{1}{r} \right\rangle = \int \frac{1}{r} \psi^* \psi$$

$$= \int_0^\infty dr \frac{1}{r} \int_0^\pi r\,d\theta \int_0^{2\pi} r \sin\theta\, d\phi \left(\frac{1}{32\pi a^3} \right) \frac{r^2}{a^2}$$

$$\exp\left(\frac{-r}{a}\right) \sin^2\theta \sin^2\phi$$

$$\left\langle \frac{1}{r} \right\rangle = \frac{1}{32\pi a^5} \underbrace{\int_0^\infty dr\, r^3 e^{-r/a}}_{6a^4} \underbrace{\int_0^{2\pi} d\phi \,\sin^2\phi}_{\pi} \underbrace{\int_0^{\pi} d\theta \,\sin^3\theta}_{4/3}$$

$$= \frac{1}{32\pi a^5} \cdot 6a^4 \cdot \pi \cdot 4/3 = \frac{1}{4a}$$

<u>Bohr Radius</u>

$$a = \frac{4\pi\varepsilon_0 \hbar^2}{me^2}$$

$$\frac{e^2}{4\pi\varepsilon_0} = \frac{\hbar^2}{ma}$$

$$\langle V \rangle = \frac{-e^2}{4\pi\varepsilon_0}\left\langle \frac{1}{r} \right\rangle = -\frac{\hbar^2}{ma}\left\langle \frac{1}{r} \right\rangle$$

$$= \frac{-\hbar^2}{4ma^2} = \frac{E_1}{2} = -6.8\,\text{eV}$$

4.38

$$V(r) = \frac{1}{2} m\omega^2 (x^2 + y^2 + z^2)$$

$$= \frac{1}{2} m\omega^2 x^2 + \frac{1}{2} m\omega^2 y^2 + \frac{1}{2} m\omega^2 z^2$$

The solutions in each dimension is a simple harmonic oscillator.

$$E = E_x + E_y + E_z$$

$$= \hbar\omega\left(n+\frac{1}{2}\right) + \hbar\omega\left(m+\frac{1}{2}\right) + \hbar\omega\left(l+\frac{1}{2}\right)$$

$$= \hbar\omega\left(n+m+l+3/2\right) = \hbar\omega\left(n'+3/2\right)$$

$$n' = n+m+l$$