# PHYS 4073 - Quantum Mechanics- Homework Set 9

**Reading Assignment: Chapter 4 Section 3 and 4**

Due at 5:45pm Monday November 15th in my box or at my office.

## Griffiths' Problems

4.19

~~4.22~~

4.23

4.44

4.48

**A1** Calculate the wavelength of the transition from the first excited state to the ground state of muonic hydrogen, a hydrogen atom where a muon has replaced the electron. Compare this value with the value you get when you incorrectly fail to use the reduced mass. Just for laughs, report the lifetime of such an atom.

**A2** Calculate the energies of the lowest five energy states of an infinite spherical well of radius $a$ in terms of the energy of the ground state of the well, $E_0$. Report the degeneracy of each state.

**A3** Construct a normalized $\psi_{744}$ wave function for the hydrogen atom.

**A4** For an atom in the $\psi_{432}$, what energies, total angular momentum, and $z$-component of angular momentum can be observed? What is the expected distance of the electron from the origin and the uncertainty in the expected distance from the origin?

**A5** A hydrogen atom is in the state

$$\psi = Ar^2 e^{-r/a} \cos(\theta)$$

where $A$ is a constant to be determined by normalization. What is the lowest energy that can be observed for this state? What is the probability of observing this energy? What angular momentum (total and $z$ components) can be observed with what probability?

25 points each

(4.19) (a)

$$[L_z, X] = [XP_y - YP_z, X]$$

$$= \overset{0}{\overset{\shortparallel}{[XP_y, X]}} - [YP_x, X]$$

$$= -Y[P_x, X] = i\hbar Y$$

$$[L_z, Y] = [XP_y - YP_x, Y]$$

$$= [XP_y, Y] - \overset{0}{\overset{\shortparallel}{[YP_x, Y]}}$$

$$= X[P_y, Y] = -i\hbar X$$

$$[L_z, Z] = [XP_y - YP_x, Z] = 0$$

$Z$ commutes with all components.

$$[L_z, P_x] = [XP_y - YP_x, P_x]$$

$$= [XP_y, P_x] - \overbrace{[YP_x, P_x]}^{=0}$$

$$= P_y [X, P_x] = i\hbar P_y$$

$$[L_z, P_y] = [XP_y - YP_x, P_y]$$

$$= \overbrace{[XP_y, P_y]}^{=0} - [YP_x, P_y]$$

$$= -P_x [Y, P_y] = -i\hbar P_x$$

$$[L_z, P_z] = [XP_y - YP_x, P_z] = 0$$

$P_z$ commutes with everything

(b)

$$[L_z, L_x] = [L_z, YP_z - ZP_y]$$

$$= [L_z, YP_z] - [L_z, ZP_y]$$

$L_z$ commutes with $P_z, Z$ as does $Y, P_y$

$$[L_z, L_x] = P_z[L_z, Y] - [L_z, P_y]Z$$

$$= P_z(-i\hbar X) - (-i\hbar P_x)Z$$

$$= i\hbar[ZP_x - P_zX]$$

$$= i\hbar L_y$$

(c)

$$[L_z, R^2] = [L_z, X^2] + [L_z, Y^2] + \overset{0}{\overset{\shortparallel}{[L_z, Z^2]}}$$

$$= -[X^2, L_z] - [Y^2, L_z]$$

$$= -X[X, L_z] - [X, L_z]X \qquad 3.13$$

$$-Y[Y, L_z] - [Y, L_z]Y$$

$$= X[L_z, X] + [L_z, X]X$$

$$+ Y[L_z, Y] + [L_z, Y]Y$$

$$= X\, i\hbar\, Y + i\hbar\, Y X$$

$$Y(-i\hbar X) - i\hbar X Y = 0$$

(b)

$$
\begin{aligned}
[L_z, P^2] &= [L_z, P_x^2] + [L_z, P_y^2] + \overbrace{[L_z, P_z^2]}^{0} \\
&= -[P_x^2, L_z] - [P_y^2, L_z] \\
&= -P_x[P_x, L_z] - [P_x, L_z]P_x \\
&\quad - P_y[P_x, L_z] - [P_y, L_z]P_y \\
&= P_x[L_z, P_x] + [L_z, P_x]P_x \\
&\quad + P_y[L_z, P_y] + [L_z, P_y]P_y \\
&= P_x(i\hbar P_y) + i\hbar P_y P_x \\
&\quad P_y(-i\hbar P_x) - i\hbar P_x P_y = 0
\end{aligned}
$$

(d) There is nothing special about the $z$ direction. Since $P^2$, $R^2$ are radial, $L_x$ and $L_y$ must also commute with $P^2$, $R^2$.

Since $H = \dfrac{P^2}{2m} + V(R)$, it must also commute with the three components of angular momentum.

4.23

$$Y_2^1 = -\left(\frac{15}{8\pi}\right)^{1/2} \sin\theta \cos\theta \, e^{i\phi}$$

$$L_+ Y_\ell^m = \hbar\sqrt{\ell(\ell+1) - m(m+1)}\; Y_\ell^{m+1}$$

$$L_+ Y_2^1 = \hbar\sqrt{2(2+1) - 1(1+1)}\; Y_2^2 = 2\hbar Y_2^2$$

$$L_+ = \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta \frac{\partial}{\partial\phi}\right)$$

$$L_+ Y_2^1 = -\hbar\left(\frac{15}{8\pi}\right)^{1/2} e^{i\phi}\left(e^{i\phi}\frac{\partial}{\partial\theta}\sin\theta\cos\theta + i\cot\theta \sin\theta\cos\theta \frac{\partial e^{i\phi}}{\partial\phi}\right)$$

$$= -\hbar\left(\frac{15}{8\pi}\right)^{1/2} e^{i\phi}\left(e^{i\phi}(\cos^2\theta - \sin^2\theta) + e^{i\phi}\, i^2 \underbrace{\cot\theta \sin\theta\cos\theta}_{\cos^2\theta}\right)$$

$$= -\hbar\left(\frac{15}{8\pi}\right)^{1/2} e^{2i\phi}\left(\cos^2\theta - \sin^2\theta - \cos^2\theta\right)$$

$$= +\hbar\left(\frac{15}{8\pi}\right)^{1/2} \sin^2\theta \, e^{2i\phi} = 2\hbar Y_2^2$$

$$Y_{-2}^{2} = \frac{1}{2}\left(\frac{15}{8\pi}\right)^{1/2} \sin^2\theta \, e^{2i\phi}$$

(4.44) $\psi_{433} = R_{43} Y_3^3$

$$= \frac{1}{768\sqrt{35}} a^{-3/2} \left(\frac{r}{a}\right)^3 \exp\left(-\frac{r}{4a}\right) \cdot$$

$$-\left(\frac{35}{64\pi}\right)^{1/2} \sin^3\theta \, e^{3i\phi}$$

$$= -\frac{1}{6144} \frac{1}{\sqrt{\pi}\, a^{9/2}} r^3 \sin^3\theta \, e^{3i\phi} e^{-r/4a}$$

(b)

$$\langle r \rangle = \int_0^\infty r \, dr \, R_{43}^* R_{43} r^2$$

$$= \frac{1}{(768)^2 35 a^9} \int_0^\infty dr \, r^9 e^{-r/2a}$$

$$= 18a$$

(c) $L_x^2 + L_y^2 = L^2 - L_z^2$

$$\langle L^2 \rangle = l(l+1)\hbar^2 = 3(3+1)\hbar^2 = 12\hbar^2$$

$$\langle L_z^2 \rangle = (m\hbar)^2 = 9\hbar^2$$

$$\langle L_x^2 + L_y^2 \rangle = 12\hbar^2 - 9\hbar^2 = 3\hbar^2$$

which will be observed 100% of the time since

~~$L_z$~~ $Y_3^3$ is an eigenstate.

Problem 4.13

*assume*( $a$ :: *real* )

*assume*( $a$ :: *positive* )

$$int\left(r^6 exp\left(\frac{-r}{a}\right),\ r = 0 ..infinity\right)$$

$$720\, a\sim^{7} \quad (1)$$

$$int\left(cos(theta)^2,\ theta = 0 ..2 \cdot Pi\right)$$

$$\pi \quad (2)$$

$$int\left(sin(theta)^5,\ theta = 0 ..Pi\right)$$

$$\frac{16}{15} \quad (3)$$

$$\frac{720 \cdot 16}{(64 \cdot 15)}$$

$$12 \quad (4)$$

Problem 4.15

$$int\left(r^3 exp\left(\frac{-r}{a}\right),\ r = 0 .. \infty\right)$$

$$6\, a\sim^{4} \quad (5)$$

$$int\left(sin(\theta)^3,\ \theta = 0 ..\pi\right)$$

$$\frac{4}{3} \quad (6)$$

Probelm 4.44

$$int\left(r^9 exp\left(\frac{-r}{2 \cdot a}\right),\ r = 0 .. \infty\right)$$

$$371589120\, a\sim^{10} \quad (7)$$

$$\frac{\%}{\left(768 \cdot 768 \cdot 35 \cdot a^9\right)}$$

$$18\, a\sim \quad (8)$$

4.48 $\hat{A} = \hat{X}^2, \quad \hat{B} = L_z$

$$[\hat{X}^2, L_z] = [X^2, XP_y - YP_x]$$

$$= \underbrace{[X^2, XP_y]}_{=0} - [X^2, YP_x]$$

$$= -Y[X^2, P_x] = -\hat{Y}2i\hbar\hat{X} \qquad (3.13)$$

since $Y$ commutes with $X, P_x$

Uncertainty Principle

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i}\langle[\hat{A}, \hat{B}]\rangle\right)^2$$

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \cdot \langle\langle -2i\hbar \hat{X}\hat{Y}\rangle\rangle\right.$$

$$\sigma_A^2 \sigma_B^2 \geq \hbar^2 \langle \hat{X}\hat{Y}\rangle^2$$

(b) $\psi_{n\ell m}$ is an eigenstate of $L_z$, so

$$\sigma_B = 0.$$

(c) Since $\sigma_A^2 \sigma_B^2 \geq \hbar^2 \langle XY \rangle^2$

and $\sigma_B = 0 \implies \langle XY \rangle = 0.$