A1 Rydberg Formula

$$\frac{1}{\lambda} = R\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

$$R = \frac{m}{4\pi c\hbar^3}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 = 1.097\times10^{+7}\,m^{-1}$$

With reduced mass,

$$\frac{1}{\lambda} = \frac{\mu}{m_e} R\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

$$\mu = \cancel{m_\mu + m_p} \quad \frac{m_\mu m_p}{m_\mu + m_p}$$

$\cancel{m_p = 1.67\times10^{-27}\,kg}$

$\cancel{m_\mu = 1.883\times10^{-28}\,kg}$

$m_p = 1.67262\times10^{-27}\,kg$

$m_\mu = 1.8835\times10^{-28}\,kg$

$$= 1.693\times10^{-28}\,kg$$

$$\frac{1}{\lambda} = \frac{1.693\times10^{-28}\,kg}{9.11\times10^{-31}\,kg} \cdot 1.097\times10^{7}\,m^{-1}\left(\frac{1}{1}-\frac{1}{2^2}\right)$$

$$= 1.529\times10^{9}\,m^{-1}$$

$$\lambda = 0.654\ nm$$

Now if muon mass is used

$$\frac{1}{\lambda} = \frac{m_\mu}{m_e} R\left(\frac{1}{1}-\frac{1}{2^2}\right)$$

$$= \frac{1.8835\times10^{-28}\,kg}{9.11\times10^{-31}\,kg} \cdot 1.097\times10^{7}\,m^{-1} \cdot 3/4$$

$$= 1.701\times10^{9}\,m^{-1}$$

$$\lambda = 0.59\times10^{-9}\,m = 0.59\,nm$$

Lifetime of muon, $2\times10^{-6}\,s$

The NIST Reference on Constants, Units, and Uncertainty

Fundamental Physical Constants

Constants Topics:

Values

Energy Equivalents

Searchable Bibliography

Background

Constants Bibliography

Constants, Units & Uncertainty home page

muon mass

m_μ

Value	$1.883\ 531\ 30 \times 10^{-28}$ kg
Standard uncertainty	$0.000\ 000\ 11 \times 10^{-28}$ kg
Relative standard uncertainty	5.6×10^{-8}
Concise form	$1.883\ 531\ 30(11) \times 10^{-28}$ kg

Click here for **correlation coefficient** of this constant with other constants

Source: 2006 CODATA recommended values

Definition of uncertainty

Correlation coefficient with any other constant

A2 The zeros of the spherical Bessel function j_ℓ are the same as the zeros of the normal Bessel function $J_{\ell+1/2}$.

Label zeros $\beta_{n\ell}$ $\quad n = \#$ of zero, 1, 2 ...

From table in A+S

$\ell = 0$, $\quad E_{n0} = \dfrac{\hbar^2}{2ma^2} n^2\pi^2 \Rightarrow \beta_{n0} = n\pi$

$= 3.14, 6.28, 9.42$

$\ell = 1$, $\quad E_{n1} = \dfrac{\hbar^2}{2ma^2} \beta_{n1}^2$, $\quad \beta_{n1} = 4.49, 7.73, 10.90$

$\ell = 2$, $\quad E_{n2} = \dfrac{\hbar^2}{2ma^2} \beta_{n2}^2$, $\quad \beta_{n2} = 5.76, 9.10$

$\ell = 3$, $\quad E_{n3} = \dfrac{\hbar^2}{2ma^2} \beta_{n3}^2$, $\quad \beta_{n3} = 6.98, 10.4$

l	B	E	Degeneracy
0	3.14	E_{10}	1
1	4.49	$2.0\ E_{10}$	3
2	5.76	$3.4\ E_{10}$	5
1	6.28	$4\ E_{10}$	1
3	6.98	$4.9\ E_{10}$	7

Let $E_{10} = \dfrac{\hbar^2 \pi^2}{2ma^2}$

(A3) $\psi_{744} = R_{74} Y_4^4$

Construct R_{74}

$$L_2^9 = L_{n-\ell-1}^{2\ell+1} \qquad \ell = 4 \quad n = 7$$

$$L_2^k = \frac{1}{2}\left(x^2 - 2(k+2)x + (k+1)(k+2)\right)$$

Mathworld

$$L_2^9 = \frac{1}{2}\left(x^2 - 2(11)x + (10)(11)\right)$$

$$= \frac{1}{2}\left(x^2 - 22x + 110\right)$$

$$R_{n\ell} = \left(\sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-\ell-1)!}{2n[(n+\ell)!]^3}}\right) e^{-r/na} \left(\frac{2r}{na}\right)^\ell \cdot L_{n-\ell-1}^{2\ell+1}\left(\frac{2r}{na}\right)$$

WolframAlpha computational knowledge engine

LaguerreL[2,9, x]

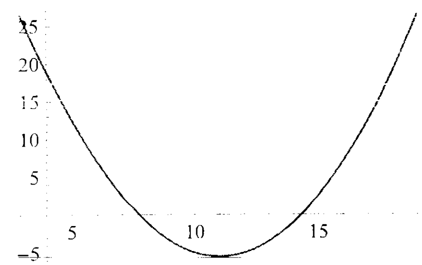
Input:

$$L_2^9(x)$$

Result:

$$\frac{1}{2}\left(x^2 - 22\,x + 110\right)$$

Plots:



(x from 5 to 17.5)

Geometric figure:

parabola

Alternate forms:

$$\frac{1}{2}(x-22)\,x + 55$$

Construct R_{74}

$$n = 7, \quad l = 4$$

$$n = j_{max} + l + 1$$

$$7 = j_{max} + 4 + 1$$

$$j_{max} = 2$$

Recurrence relation

$$a_{j+1} = \frac{2(j+l+1) - 2n}{(j+1)(j+2l+2)} a_j$$

$$a_1 = \frac{2(0+4+1) - 2 \cdot 7}{(0+1)(0+2 \cdot 4+2)} a_0$$

$$= \frac{10 - 14}{10} a_0 = -\frac{4}{10} a_0$$

$$a_2 = \frac{2(1+4+1) - 2\cdot 7}{(1+1)(1+2\cdot 4+2)}\, a_1$$

$$= \frac{12 - 14}{22}\, a_1 = -\frac{1}{11}\, a_1$$

$$= \frac{4}{110}\, a_0$$

$$V_{74}(\rho) = a_0\left(\frac{4}{110}\rho^2 - \frac{4}{10}\rho + 1\right)$$

$$= 4\rho^2 - 44\rho + 110$$

Check Against Laguerre

$$L^{2l+1}_{n-l-1} \stackrel{?}{=} L^{9}_{2}$$

$$L^{p}_{q-p} \qquad p = 9 \qquad q = 11$$

$$L_q = e^{x} \left(\frac{d}{dx} \right) \left(e^{-x} x^{q} \right)$$

$$L^{p}_{q-p} = (-1)^{p} \left(\frac{d}{dx} \right)^{p} L_q(x)$$

I constructed L^9_2 with Maxima

$$L^{9}_{2} = A \left(x^2 - 22x + 110 \right)$$

So the term in R_{74} is

$$L^{9}_{2}(2\rho) = A \left(4\rho^2 - 44\rho + 110 \right)$$

which checks.

```
(%i1) diff(x^2, x, 2);
```

(%o1) 2

```
(%i2) diff(exp(2*x), x);
```

(%o2) $2\,\%e^{2\,x}$

```
(%i3) l11:exp(x)*diff(exp(-x)*x^11, x, 11);
```

(%o3) $(-x^{11}\,\%e^{-x}+121\,x^{10}\,\%e^{-x}-6050\,x^{9}\,\%e^{-x}+163350\,x^{8}\,\%e^{-x}-2613600\,x^{7}\,\%e^{-x}+25613280\,x^{6}\,\%e^{-x}-153679680\,x^{5}\,\%e^{-x}+548856000\,x^{4}\,\%e^{-x}-1097712000\,x^{3}\,\%e^{-x}+1097712000\,x^{2}\,\%e^{-x}-439084800\,x\,\%e^{-x}+39916800\,\%e^{-x})\,\%e^{x}$

```
(%i4) assocl92:(-1)^9*diff(l11, x, 9);
```

(%o4) $-9\,(x^{11}\,\%e^{-x}-132\,x^{10}\,\%e^{-x}+7260\,x^{9}\,\%e^{-x}-217800\,x^{8}\,\%e^{-x}+3920400\,x^{7}\,\%e^{-x}-43908480\,x^{6}\,\%e^{-x}+307359360\,x^{5}\,\%e^{-x}-1317254400\,x^{4}\,\%e^{-x}+3293136000\,x^{3}\,\%e^{-x}-4390848000\,x^{2}\,\%e^{-x}+2634508800\,x\,\%e^{-x}-479001600\,\%e^{-x})\,\%e^{x}$
$-84\,(x^{11}\,\%e^{-x}-154\,x^{10}\,\%e^{-x}+10010\,x^{9}\,\%e^{-x}-360360\,x^{8}\,\%e^{-x}+7927920\,x^{7}\,\%e^{-x}-110990880\,x^{6}\,\%e^{-x}+998917920\,x^{5}\,\%e^{-x}-5708102400\,x^{4}\,\%e^{-x}+19978358400\,x^{3}\,\%e^{-x}-39956716800\,x^{2}\,\%e^{-x}+39956716800\,x\,\%e^{-x}-14529715200\,\%e^{-x})\,\%e^{x}$
$-126\,(x^{11}\,\%e^{-x}-176\,x^{10}\,\%e^{-x}+13200\,x^{9}\,\%e^{-x}-554400\,x^{8}\,\%e^{-x}+14414400\,x^{7}\,\%e^{-x}-242161920\,x^{6}\,\%e^{-x}+2663781120\,x^{5}\,\%e^{-x}-19027008000\,x^{4}\,\%e^{-x}+85621536000\,x^{3}\,\%e^{-x}-228324096000\,x^{2}\,\%e^{-x}+319653734400\,x\,\%e^{-x}-174356582400\,\%e^{-x})\,\%e^{x}$
$-36\,(x^{11}\,\%e^{-x}-198\,x^{10}\,\%e^{-x}+16830\,x^{9}\,\%e^{-x}-807840\,x^{8}\,\%e^{-x}+24235200\,x^{7}\,\%e^{-x}-475009920\,x^{6}\,\%e^{-x}+6175128960\,x^{5}\,\%e^{-x}-52929676800\,x^{4}\,\%e^{-x}+291113222400\,x^{3}\,\%e^{-x}-970377408000\,x^{2}\,\%e^{-x}+1746679334400\,x\,\%e^{-x}-1270312243200\,\%e^{-x})\,\%e^{x}$
$-(x^{11}\,\%e^{-x}-220\,x^{10}\,\%e^{-x}+20900\,x^{9}\,\%e^{-x}-1128600\,x^{8}\,\%e^{-x}+38372400\,x^{7}\,\%e^{-x}-859541760\,x^{6}\,\%e^{-x}+12893126400\,x^{5}\,\%e^{-x}-128931264000\,x^{4}\,\%e^{-x}+838053216000\,x^{3}\,\%e^{-x}-3352212864000\,x^{2}\,\%e^{-x}+7374868300800\,x\,\%e^{-x}-6704425728000\,\%e^{-x})\,\%e^{x}$
$-9\,(-x^{11}\,\%e^{-x}+209\,x^{10}\,\%e^{-x}-18810\,x^{9}\,\%e^{-x}+959310\,x^{8}\,\%e^{-x}-30697920\,x^{7}\,\%e^{-x}+644656320\,x^{6}\,\%e^{-x}-9025188480\,x^{5}\,\%e^{-x}+83805321600\,x^{4}\,\%e^{-x}-502831929600\,x^{3}\,\%e^{-x}+1843717075200\,x^{2}\,\%e^{-x}-3687434150400\,x\,\%e^{-x}+3016991577600\,\%e^{-x})\,\%e^{x}$
$-84\,(-x^{11}\,\%e^{-x}+187\,x^{10}\,\%e^{-x}-14960\,x^{9}\,\%e^{-x}+673200\,x^{8}\,\%e^{-x}-18849600\,x^{7}\,\%e^{-x}+343062720\,x^{6}\,\%e^{-x}-4116752640\,x^{5}\,\%e^{-x}+32345913600\,x^{4}\,\%e^{-x}-161729568000\,x^{3}\,\%e^{-x}+485188704000\,x^{2}\,\%e^{-x}-776301926400\,x\,\%e^{-x}+494010316800\,\%e^{-x})\,\%e^{x}$
$-126\,(-x^{11}\,\%e^{-x}+165\,x^{10}\,\%e^{-x}-11550\,x^{9}\,\%e^{-x}+450450\,x^{8}\,\%e^{-x}-10810800\,x^{7}\,\%e^{-x}+166486320\,x^{6}\,\%e^{-x}-1664863200\,x^{5}\,\%e^{-x}+10702692000\,x^{4}\,\%e^{-x}-42810768000\,x^{3}\,\%e^{-x}+99891792000\,x^{2}\,\%e^{-x}-119870150400\,x\,\%e^{-x}+54486432000\,\%e^{-x})\,\%e^{x}$
$-36\,(-x^{11}\,\%e^{-x}+143\,x^{10}\,\%e^{-x}-8580\,x^{9}\,\%e^{-x}+283140\,x^{8}\,\%e^{-x}-5662800\,x^{7}\,\%e^{-x}+71351280\,x^{6}\,\%e^{-x}-570810240\,x^{5}\,\%e^{-x}+2854051200\,x^{4}\,\%e^{-x}-8562153600\,x^{3}\,\%e^{-x}+14270256000\,x^{2}\,\%e^{-x}-11416204800\,x\,\%e^{-x}+3113510400\,\%e^{-x})\,\%e^{x}$
$-(-x^{11}\,\%e^{-x}+121\,x^{10}\,\%e^{-x}-6050\,x^{9}\,\%e^{-x}+163350\,x^{8}\,\%e^{-x}-2613600\,x^{7}\,\%e^{-x}+25613280\,x^{6}\,\%e^{-x}-153679680\,x^{5}\,\%e^{-x}+548856000\,x^{4}\,\%e^{-x}-1097712000\,x^{3}\,\%e^{-x}+1097712000\,x^{2}\,\%e^{-x}-439084800\,x\,\%e^{-x}+39916800\,\%e^{-x})\,\%e^{x}$

```
(%i6) factor(assocl92);
```

(%o6) $19958400\,(x^{2}-22\,x+110)$

The formula in the book depends of a standard of normalization which is not consistant across sources. Throw away constants and do normalization ourselves.

$$R_{74} = A r^4 e^{-r/7a} \cdot \left(\left(\frac{2r}{7a}\right)^2 - 22\left(\frac{2r}{7a}\right) + 110\right)$$

$$1 = \int_0^\infty dr\, r^2 R_{74}^* R_{74}$$

$$R_{74} = \frac{1}{a^{11/2}} \cdot \sqrt{\frac{2}{2.16 \times 10^{15}}}\, r^4 e^{-r/7a} \cdot \left(\frac{4r^2}{49a^2} - \frac{22r}{7a} + 110\right)$$

$$\psi_{744} = R_{74}\, \cancel{Y_{44}}\, Y_4^4$$

$$Y_4^4 = \frac{3}{16}\left(\frac{35}{2\pi}\right)^{1/2} \sin^4\theta\, e^{4i\phi}$$

Total Wave Function

$$\psi_{744} = \frac{1}{a^{11/2}} \sqrt{\frac{2}{2.16 \times 10^{15}}}\, r^4 e^{-r/7a} \cdot$$

$$\left(\frac{4r^2}{49a^2} - \frac{22r}{7a} + 110 \right) \cdot$$

$$\frac{3}{16} \left(\frac{35}{2\pi} \right)^{1/2} \sin^4\theta \, e^{4i\phi}$$

WolframAlpha

integrate r^10*e^(-2r/(7a))*((2r/(7a))^2-44r/(7a)+110)^2 r=0 to infinity

Definite integral:

$$\int_0^\infty r^{10} e^{-\frac{2r}{7a}} \left(\left(\frac{2r}{7a} \right)^2 - \frac{44r}{7a} + 110 \right)^2 dr = \frac{2\,158\,202\,706\,815\,925\,a^{11}}{2} \quad \text{if } \operatorname{Re}(a) > 0$$

A4

$$R_{43} = \frac{1}{768\sqrt{35}} a^{-3/2} \left(\frac{r}{a}\right)^3 e^{-r/4a}$$

Table 4.7

$$Y_3^2 = \left(\frac{105}{32\pi}\right)^{1/2} \sin^2\theta \cos\theta \, e^{2i\phi}$$

$$\psi_{432} = R_{43} Y_3^2 = \frac{1}{768\sqrt{35}} a^{-3/2} \left(\frac{r}{a}\right)^3 e^{-r/4a}$$

$$\left(\frac{105}{32\pi}\right)^{1/2} \sin^2\theta \cos\theta \, e^{2i\phi}$$

The energy is $E_4 = \frac{-13.6\,eV}{4^2} = -0.85\,eV$ $\quad n=4$

$$L = \sqrt{l(l+1)\hbar^2} = 12\hbar \qquad l=3$$

$m=2$

$$L_z = 2\hbar$$

with 100% probability.

$$\langle r \rangle = \int \psi^* \psi \, r \left(r^2 \sin\theta \, dr \, d\theta \, d\phi \right)$$

$$= \int_0^\infty r \, R^* R \, r^2 \, dr \qquad (Y \text{ normalized separately})$$

$$= \frac{1}{768^2 \cdot 35} \, \frac{1}{a^9} \underbrace{\int_0^\infty r^9 e^{-r/2a} \, dr}_{371,589,120 \, a^{10}}$$

* alpha

$$= \frac{371,589,120}{768^2 \cdot 35} \, a = 18a$$

$$\langle r^2 \rangle = \int_0^\infty r^2 \, R^* R \, r^2 \, dr$$

$$= \frac{1}{768^2 \cdot 35} \, \frac{1}{a^9} \int_0^\infty r^{10} e^{-r/2a} \, dr$$

$$= \frac{1}{768^2 \cdot 35} \cdot \frac{1}{a^9} \cdot 7431782400 \, a^{11}$$

$$= 360 \, a^2$$

$$\sigma_r = \sqrt{\langle r^2 \rangle - \langle r \rangle^2}$$

$$= \sqrt{360a^2 - 18^2 a^2}$$

$$= 6a$$

WolframAlpha

integrate r^(10) exp(−r/2a) from r=0 to infinity

Definite integral:

$$\int_0^\infty r^{10} \exp\left(-\frac{r}{2\,a}\right) d\,r = 7\,431\,782\,400\,a^{11} \quad \text{for } \operatorname{Re}(a) > 0$$

WolframAlpha computational knowledge engine

integrate r^9 exp(−r/2a) from r=0 to infinity

Definite integral:

$$\int_0^\infty r^9 \exp\left(-\frac{r}{2a}\right) dr = 371\,589\,120\,a^{10}$$

A5 $\psi = A r^2 e^{-r/a} \cos\theta$

Comparison of angular part with the spherical harmonics shows $\psi = R(r)\, Y_1^0$

$$Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta$$

$$R = A' r^2 e^{-r/a}$$

Note, R is not a pure eigenstate so

$$R(r) = \sum_{n=2} c_n R_{n1}(r)$$

<u>First Normalized</u>

$$1 = \int_0^\infty dr\, r^2 R^* R = A'^2 \int_0^\infty dr\, r^6 e^{-2r/a}$$

$$= A'^2 \frac{47 a^7}{8}$$

$$A' = \sqrt{\frac{8}{47}}\, \frac{1}{a^{7/2}}$$

WolframAlpha computational knowledge engine

integrate r^6 e^(−2r/a) from r=0 to infinity

Definite integral:

$$\int_0^\infty r^6 e^{-\frac{2r}{a}}\, dr = \frac{45\, a^7}{8} \quad \text{if } \mathrm{Re}(a) > 0$$

$$\psi = \sqrt{\frac{8}{47}} \frac{1}{a^{7/2}} r^2 e^{-r/a} Y_1^0$$

The lowest energy state with $\ell = 1$ has radial function R_{21} for $n = 2$.

From table,

$$R_{21} = \frac{1}{\sqrt{24}} a^{-3/2} \frac{r}{a} \exp\left(\frac{-r}{2a}\right)$$

Use Fourier's Trick $R_{n\,\ell=1}$ orthogonal

$$C_2 = \int_0^\infty dr\, r^2 R_{21}^* R$$

$$= \sqrt{\frac{1}{24}} \sqrt{\frac{8}{47}} \frac{1}{a^6} \int_0^\infty r^5 e^{-\frac{3r}{2a}} dr$$

$$= \sqrt{\frac{1}{24}} \sqrt{\frac{8}{47}} \frac{2560}{243} \frac{a^6}{a^6}$$

$$= 0.8872$$

The probability to observe the system in the $n=2$ $\ell=1$ $m=0$ state

is $c_2^* c_2 = 0.79$