Identical Particles

Consider an operation that exchanges two particles in a multi-particle wave function

$$P_{12}\,\psi(\vec{r}_1,\vec{r}_2) = \psi(\vec{r}_2,\vec{r}_1)$$

If we can tell the particles apart, that is if they are <u>distinguishable</u>, there is no simple relation between $\psi(\vec{r}_1,\vec{r}_2)$ and $\psi(\vec{r}_2,\vec{r}_1)$.

If the particles are identical, that is if no experiment can tell them apart, then exchanging the particles cannot change experimental results. Experimental results are determined by the probability density

$$\mathbb{P}(\vec{r}_1,\vec{r}_2) = \psi^*(\vec{r}_1,\vec{r}_2)\,\psi(\vec{r}_1,\vec{r}_2)$$

For identical particles,

$$P(\vec{r}_1, \vec{r}_2) = P(\vec{r}_2, \vec{r}_1)$$

$$\Rightarrow \quad \psi(\vec{r}_1, \vec{r}_2) = \pm\psi(\vec{r}_2, \vec{r}_1)$$

<u>Symmetrization Requirement</u> (Add to Postulates)

For identical particles, the wave function must be unchanged up to a sign by pairwise exchange of particles.

$$\psi(\vec{r}_1, \vec{r}_2, \cdots \vec{r}_n) = \pm\psi(\vec{r}_2, \vec{r}_1, \cdots, \vec{r}_n)$$

The sign is determined by the spin of the particle.

$+$ Bosons - Integer Spin - $\psi(\vec{r}_1, \vec{r}_2) = \psi(\vec{r}_2, \vec{r}_1)$

$-$ Fermions - Half-Integer Spin

$$\psi(\vec{r}_1, \vec{r}_2) = -\psi(\vec{r}_2, \vec{r}_1)$$

Return to our non-interacting two particle system

$$\psi = \psi_1^a(x_1)\,\psi_2^b(x_2)$$

is no longer a valid wave function if $a \neq b$.

<u>For bosons</u> To be unchange under particle exchange

$$\psi = \frac{1}{\sqrt{2}}\left(\psi_1^a(x_1)\,\psi_2^b(x_2) + \psi_2^a(x_2)\,\psi_1^b(x_1)\right)$$

<u>For Fermions,</u> To change sign under particle exchange

$$\psi = \frac{1}{\sqrt{2}}\left(\psi_1^a(x_1)\,\psi_2^b(x_2) - \psi_2^a(x_2)\,\psi_1^b(x_1)\right)$$

$\Longrightarrow$ Note, for Fermions $\psi = 0$ for $a = b$

Pauli Exclusion Principle

No two fermions can occupy the same state.

The symmetrization requirement generates a fictitious force between identical particles that causes attraction for bosons and repulsion for fermions. This force is called an exchange force.

Consider our three one-dimensional two particle wave functions, $a \neq b$

Distinguishable $\psi = \psi_1^a(x_1)\, \psi_2^b(x_2)$

Boson $\psi = \frac{1}{\sqrt{2}}\left(\psi_1^a\, \psi_2^b + \psi_2^a\, \psi_1^b \right)$

Fermion $\psi = \frac{1}{\sqrt{2}}\left(\psi_1^a\, \psi_2^b - \psi_2^a\, \psi_1^b \right)$

The average square distance between the particles is

$$\langle (x_1 - x_2)^2 \rangle = \langle x_1^2 \rangle + \langle x_2^2 \rangle - 2 \langle x_1 x_2 \rangle$$

<u>Distinguishable</u>

$$\langle x_1^2 \rangle = \int dx_1 \int dx_2 \, x_1^2 \, \psi_1^{a*} \psi_1^a \, \psi_2^{b*} \psi_2^b$$

$$= \int dx_1 \, x_1^2 \, \psi_1^{a*} \psi_1^a \equiv \langle x_1^2 \rangle$$

$$= \int dx \, x^2 \, \psi^{a*}(x) \psi^a(x) \equiv \langle x^2 \rangle^a$$

<u>Likewise</u>

$$\langle x_2^2 \rangle = \langle x^2 \rangle^b = \int dx_1 \int dx_2 \, x_2^2 \, \psi_1^{a*} \psi_1^a \, \psi_2^{b*} \psi_2^b$$

$$= \int dx_2 \, x_2^2 \, \psi_2^{b*} \psi_2^b = \int dx \, x^2 \, \psi^b(x)^* \psi^b(x)$$

$$= \langle x^2 \rangle^b$$

And the cross-term

$$\langle x_1 x_2 \rangle = \int dx_1 \int dx_2 \, x_1 x_2 \, \psi_1^{a*} \psi_1^a \, \psi_2^{b*} \psi_2^b$$

$$= \left[\int dx_1 x_1 \, \psi_1^{a*} \psi_1^a \right] \left[\int dx_2 x_2 \, \psi_2^{b*} \psi_2^b \right]$$

$$\langle x_1 x_2 \rangle = \langle x \rangle^a \langle x \rangle^b$$

So if the particles are distinguishable

$$\langle (x_1 - x_2)^2 \rangle = \langle x^2 \rangle^a + \langle x^2 \rangle^b - 2 \langle x \rangle^a \langle x \rangle^b$$

For Bosons

$$\psi^* \psi = \frac{1}{2} \left[\psi_1^{a*} \psi_2^{b*} + \psi_2^{a*} \psi_1^{b*} \right] \left[\psi_1^a \psi_2^b + \psi_2^a \psi_1^b \right]$$

$$= \frac{1}{2} \Big[\underset{1}{\psi_1^{a*} \psi_2^{b*} \psi_1^a \psi_2^b} + \underset{2}{\psi_1^{a*} \psi_2^{b*} \psi_2^a \psi_1^b}$$

$$+ \underset{3}{\psi_2^{a*} \psi_1^{b*} \psi_1^a \psi_2^b} + \underset{4}{\psi_2^{a*} \psi_1^{b*} \psi_2^a \psi_1^b} \Big]$$

For $\langle x_1^2 \rangle$

1 $= \langle x^2 \rangle^a$ 2 $= 0$ (particle 2 orthogonal)

3 $= 0$ 4 $= \langle x^2 \rangle^b$

For $\langle x_2^2 \rangle$

$$① = \langle x^2 \rangle^b \qquad ② = 0$$

$$③ = 0 \qquad ④ = \langle x^2 \rangle^a$$

For $\langle x_1 x_2 \rangle$

$$① = \langle x \rangle^a \langle x \rangle^b$$

$$② = \left(\int dx_1\, x_1\, \psi_1^{a*} \psi_1^{b} \right) \left(\int dx_2\, x_2\, \psi_2^{b*} \psi_2^{a} \right)$$

Define $\displaystyle\int dx\, x\, \psi^{a*}(x)\, \psi^b(x) = \langle x \rangle^{ab}$

so

$$② = \left| \langle x \rangle^{ab} \right|^2$$

$$③ = \left| \langle x \rangle^{ab} \right|^2$$

$$④ = \langle x \rangle^a \langle x \rangle^b$$

For bosons

$$\langle (x_1 - x_2)^2 \rangle = \langle x^2 \rangle^a + \langle x^2 \rangle^b - 2\langle x \rangle^a \langle x \rangle^b$$

$$- 2\,|\langle x \rangle^{ab}|^2$$

closer than distinguishable particles.

For Fermions

Same calculation except the $|\langle x \rangle^{ab}|$ term changes sign.

$$\langle (x_1 - x_2)^2 \rangle = \langle x^2 \rangle^a + \langle x^2 \rangle^b - 2\langle x \rangle^a \langle x \rangle^b$$

$$+ 2\,|\langle x \rangle_{ab}|^2$$

$\Rightarrow$ On average farther apart for fermions

The symmetrization requirement affects allowed energy states and their degeneracy.

Consider the simple harmonic oscillator with two non-interacting particles $E^n = (n+\frac{1}{2})\hbar\omega$

Distinguishable

E	ψ	Degeneracy
$\hbar\omega$	$\psi_1^0 \psi_2^0$	1
$2\hbar\omega$	$\psi_1^0 \psi_2^1, \psi_1^1 \psi_2^0$	2
$3\hbar\omega$	$\psi_1^1 \psi_2^1$	1

Bosons

E	ψ	Degeneracy
$\hbar\omega$	$\psi_1^0 \psi_2^0$	1
$2\hbar\omega$	$\frac{1}{\sqrt{2}}\left(\psi_1^0 \psi_2^1 + \psi_2^0 \psi_1^1\right)$	1
$3\hbar\omega$	$\psi_1^1 \psi_2^1$	1

Fermions

E	ψ	Degeneracy
$2\hbar\omega$	$\frac{1}{\sqrt{2}}\left(\psi_1^0\psi_2^1 - \psi_2^0\psi_1^1\right)$	1
$3\hbar\omega$	$\frac{1}{\sqrt{2}}\left(\psi_1^0\psi_2^2 - \psi_2^0\psi_1^2\right)$	1
$4\hbar\omega$	$\frac{1}{\sqrt{2}}\left(\psi_1^1\psi_2^2 - \psi_2^1\psi_1^2\right.$	1

Slater Determinant

Constructs fermion state

$$\psi = \frac{1}{\sqrt{2}} \det \begin{pmatrix} \phi_a(x_1) & \phi_b(x_1) \\ \phi_a(x_2) & \phi_b(x_2) \end{pmatrix}$$

$$\psi = \frac{1}{\sqrt{N!}} \det \begin{pmatrix} \phi_a(x_1) & \cdots & \phi_f(x_1) \\ \phi_a(x_2) & & \vdots \\ \vdots & & \vdots \\ \phi_a(x_n) & & \phi_f(x_n) \end{pmatrix}$$

Ex Griffiths Problem 5.6

Two particles are in an infinite square well $x \in [0,a]$. Compute $\langle (x_1 - x_2)^2 \rangle$ for ψ^n, ψ^m if the particles are distinguishable, identical bosons, or fermions.

$$\psi^n = \sqrt{\frac{2}{a}} \sin \frac{n\pi}{a} x$$

$$\langle x \rangle^n = \int_0^a dx \; x \, \psi^{n*} \psi^n$$

$$= \frac{2}{a} \int_0^a x \sin^2 \frac{n\pi}{a} x \; dx$$

$$= \frac{a}{2}$$

$$\langle x^2 \rangle^n = \int_0^a dx \; x^2 \psi^{n*} \psi^n = a^2 \left(\frac{1}{3} - \frac{1}{2(n\pi)^2} \right)$$

$$\langle x \rangle^{nm} = \int_0^a dx\, x\, \psi^{n*} \psi^m$$

$$= \int_0^a dx\, x \sin \frac{n\pi}{a} x \sin \frac{m\pi}{a} x$$

$$= \frac{a}{\pi^2} \left[\frac{\cos((m-n)\pi)}{(m-n)^2} - \frac{1}{(m-n)^2} - \frac{\cos((m+n)\pi)}{(m+n)^2} + \frac{1}{(m+n)^2} \right]$$

$$= 0 \quad \text{if} \quad m+n \quad \text{even}$$

$$= \frac{a}{\pi^2} \left(-\frac{2}{(m-n)^2} + \frac{2}{(m+n)^2} \right)$$

$$= \frac{2}{\pi^2} \left(\frac{(m-n)^2 - (m+n)^2}{(m-n)^2 (m+n)^2} \right)$$

$$\langle x \rangle^{nm} = \frac{2a}{\pi}\left(\frac{m^2 - 2mn - n^2 - m^2 - 2mn - n^2}{(m^2-n^2)^2}\right)$$

$$= \frac{-8amn}{\pi^2(m^2-n^2)^2}$$

<u>Distinguishable</u>

$$\langle (x_1 - x_2)^2 \rangle^d = \langle x^2 \rangle^m + \langle x^2 \rangle^n - 2\langle x \rangle^n \langle x \rangle^m$$

$$= a^2\left(\frac{1}{3} - \frac{1}{2(m\pi)^2}\right) + a^2\left(\frac{1}{3} - \frac{1}{2(n\pi)^2}\right)$$

$$- 2\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)$$

$$= \frac{a^2}{6} - \frac{a^2}{2\pi^2}\left(\frac{1}{n^2} + \frac{1}{m^2}\right)$$

+ for fermions

Bosons

$$\langle (x_1 - x_2)^2 \rangle = \langle (x_1 - x_2)^2 \rangle^{d} - 2 \left| \langle x \rangle^{mn} \right|^2$$

$$= \frac{a}{6} - \frac{a^2}{2\pi^2}\left(\frac{1}{n^2} + \frac{1}{m^2} \right) - \frac{128\, a^2 m^2 n^2}{\pi^4 (m^2 - n^2)^2}$$

Fermion

Same except plus.