Inner Products

The dot product $\vec{A} \cdot \vec{B}$ is an example of an <u>inner product</u>, a function $I(|a\rangle, |b\rangle)$ that takes two vectors and returns a scalar.

<u>Dfn</u> Inner product, $I(|a\rangle, |b\rangle) = c$

complex number

<u>Properties</u>

① $I(|a\rangle, |b\rangle) = I^*(|b\rangle, |a\rangle)$

complex conjugate

$\Rightarrow I(|a\rangle, |a\rangle)$ real.

② $I(|a\rangle, |a\rangle) \geq 0$

③ If $I(|a\rangle, |a\rangle) = 0$, then $|a\rangle = |0\rangle$.

④ $I(|a\rangle, c_1|b\rangle + c_2|c\rangle)$

$= c_1 I(|a\rangle, |b\rangle) + c_2 I(|a\rangle, |c\rangle)$

The length or <u>norm</u> of a vector can be written $\|\vec{A}\| = \sqrt{\vec{A}\cdot\vec{A}}$

Dfn Norm Vector

$$\| |a\rangle \| = \sqrt{I(|a\rangle, |a\rangle)}$$

$\Rightarrow$ Generalized length of vector

Normalized Vector (Unit Vector) - A vector with norm = 1 can be constructed by

$$|e\rangle = \frac{|a\rangle}{\| |a\rangle \|} \quad = \text{normalized vector}$$

compare with normal unit vector

$$\hat{r} = \frac{\vec{r}}{r} = \frac{\vec{r}}{\sqrt{\vec{r}\cdot\vec{r}}}$$

Dfn <u>Orthogonal</u> Two vectors are orthogonal if

$$I(|a\rangle, |b\rangle) = 0$$

Dfn <u>Orthonormal Basis</u> A basis for a vector space $\{|e_1\rangle, |e_2\rangle \cdots\}$ is orthonormal if

$$I(|e_i\rangle, |e_j\rangle) = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

<u>Schwartz Inequality</u>

$$|I(|a\rangle, |b\rangle)|^2 \leq I(|a\rangle, |a\rangle)\, I(|b\rangle, |b\rangle)$$

<u>Dirac Notation</u>

$$I(|a\rangle, |b\rangle) \Longrightarrow \langle a|b\rangle$$

$$\Longrightarrow \langle a|b\rangle = \langle b|a\rangle^*$$

$$\Longrightarrow \langle a|a\rangle = \text{real} \geq 0$$

Gram-Schmidt Process for construction on orthonormal basis

$$\{|a_i\rangle\} \xrightarrow{GS} \{|e_i\rangle\}$$

basis $\qquad\qquad$ orthonormal basis

1. Normalize $|a_1\rangle$

$$|e_1\rangle = \frac{|a_1\rangle}{\sqrt{\langle a_1|a_1\rangle}}$$

2. Subtract component of $|a_2\rangle$ along $|a_1\rangle$.

$$|b_2\rangle \equiv |a_2\rangle - \langle e_1|a_2\rangle |e_1\rangle$$

3. Normalize $|b_2\rangle$

$$|e_2\rangle = \frac{|b_2\rangle}{\sqrt{\langle b_2|b_2\rangle}}$$

4 Subtract projection of $|a_3\rangle$ on $|e_1\rangle$, $|e_2\rangle$

$$|b_3\rangle = |a_3\rangle - \langle e_1|a_1\rangle|e_1\rangle - \langle e_2|a_2\rangle|e_2\rangle$$

5 Normalize

$$|e_3\rangle = \frac{|b_3\rangle}{\sqrt{\langle b_3|b_3\rangle}}$$

6 Keep Going

Example Construct orthonormal basis from

$$\{ |a_1\rangle = \hat{x} + \hat{y}, \quad |a_2\rangle = \hat{x} - \hat{y}, \quad |a_3\rangle = \hat{x} + \hat{y} + \hat{z} \}$$

1 $$|e_1\rangle = \frac{|a_1\rangle}{\sqrt{\langle a_1|a_1\rangle}} = \frac{\hat{x} + \hat{y}}{\sqrt{(\hat{x}+\hat{y})\cdot(\hat{x}+\hat{y})}}$$

$$= \frac{\hat{x} + \hat{y}}{\sqrt{2}}$$

(2) Project $|e_1\rangle$ out of $|a_2\rangle$

$$|b_2\rangle = |a_2\rangle - \langle e_1|a_2\rangle|e_1\rangle$$

$$= \hat{x} - \hat{y} - \left(\left(\frac{\hat{x}+\hat{y}}{\sqrt{2}}\right)\cdot(\hat{x}-\hat{y})\right)\left(\frac{\hat{x}+\hat{y}}{\sqrt{2}}\right)$$

$$= \hat{x} - \hat{y} - \frac{1}{\sqrt{2}}(0)\,\frac{\hat{x}+\hat{y}}{\sqrt{2}}$$

$$= \hat{x} - \hat{y}$$

(3) Normalize $|b_2\rangle$

$$|e_2\rangle = \frac{|b_2\rangle}{\sqrt{\langle b_2|b_2\rangle}} = \frac{\hat{x}-\hat{y}}{\sqrt{(\hat{x}-\hat{y})\cdot(\hat{x}-\hat{y})}}$$

$$= \frac{\hat{x}-\hat{y}}{\sqrt{2}}$$

4 Project $|e_1\rangle$, $|e_2\rangle$ out of $|a_3\rangle$

$$|b_3\rangle = |a_3\rangle - \langle e_1|a_3\rangle|e_1\rangle - \langle e_2|a_3\rangle|e_2\rangle$$

$$= \hat{x}+\hat{y}+\hat{z} - \left[\left(\frac{\hat{x}+\hat{y}}{\sqrt{2}}\right)\cdot\left(\hat{x}+\hat{y}+\hat{z}\right)\right]\left(\frac{\hat{x}+\hat{y}}{\sqrt{2}}\right) - \left[\left(\frac{\hat{x}-\hat{y}}{\sqrt{2}}\right)\cdot\left(\hat{x}+\hat{y}+\hat{z}\right)\right]\left(\frac{\hat{x}-\hat{y}}{\sqrt{2}}\right)$$

$$= \hat{x}+\hat{y}+\hat{z} - \frac{1}{\sqrt{2}}(2)\frac{\hat{x}+\hat{y}}{\sqrt{2}} - \frac{1}{\sqrt{2}}\cdot 0$$

$$= \hat{x}+\hat{y}+\hat{z} - (\hat{x}+\hat{y}) = \hat{z}$$

5 Normalize $|b_3\rangle$

$$|e_3\rangle = \frac{|b_3\rangle}{\sqrt{\langle b_3|b_3\rangle}} = \frac{\hat{z}}{\sqrt{\hat{z}\cdot\hat{z}}} = \hat{z}$$

The orthonormal basis is

$$\left\{ \frac{\hat{x}+\hat{y}}{\sqrt{2}} , \frac{\hat{x}-\hat{y}}{\sqrt{2}} , \hat{z} \right\}$$