Moment in Uniform Field

Magnetic Moment in Uniform Field $\vec{B} = B_0 \hat{z}$

Potential Energy $U = -\vec{\mu} \cdot \vec{B}$

$$= -\gamma \vec{S} \cdot \vec{B}$$

Hamiltonian

$$H = \frac{P^2}{2m} + U$$

$$= H_k + H_r$$

The kinetic part of the energy separates from rotational part; let's work on only the rotational part.

$$\hat{H} = -\gamma \hat{\vec{S}} \cdot \hat{\vec{B}}$$

$$= -\gamma B_0 \hat{S}_z$$

Eigenvectors of $\hat{H}$ $\quad \{|+\rangle, |-\rangle\}$

where

$$|+\rangle = |s = \tfrac{1}{2}, m = \tfrac{1}{2}\rangle$$

$$|-\rangle = |s = \tfrac{1}{2}, m = -\tfrac{1}{2}\rangle$$

Energies

$$\hat{H}|+\rangle = E_+|+\rangle = -\gamma B_0 \hat{S}_z |+\rangle$$

$$= -\frac{\gamma B_0 \hbar}{2} |+\rangle$$

$$E_+ = -\frac{\gamma B_0 \hbar}{2}$$

Likewise, $\quad E_- = \dfrac{\gamma B_0 \hbar}{2}$

<u>Time Evolution</u> The ~~basis~~ eigenstates of the Hamiltonian form a basis, so any state of the system may be written as a linear combination

$$|\psi(0)\rangle = a|+\rangle + b|-\rangle$$

or as a vector

$$|\psi(0)\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$ called a <u>spinor</u>

Since this is the energy basis, time evolution is simple

$$|\psi(t)\rangle = a e^{-i\frac{E_+ t}{\hbar}}|+\rangle + b e^{-i\frac{E_- t}{\hbar}}|-\rangle$$

$$= a e^{i\frac{\gamma B_0 t}{2}}|+\rangle + b e^{-i\frac{\gamma B_0 t}{2}}|-\rangle$$

Simplify using Larmor frequency

$$\omega = \gamma B_0$$

and take care of normalization by defining

$$a = \cos\frac{\alpha}{2} \quad \Rightarrow \quad b = \sin\frac{\alpha}{2}$$

$$1 = a^2 + b^2$$

then

$$|\psi(t)\rangle = \cos\frac{\alpha}{2}\, e^{i\omega t/2}|+\rangle + \sin\frac{\alpha}{2}\, e^{-i\omega t/2}|-\rangle$$

We've got ψ, let's compute some expectation values.

$$S^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\langle \psi | S^2 | \psi \rangle =$$

$$\left(\cos\frac{\alpha}{2} e^{-i\omega t/2}, \sin\alpha/2\, e^{i\omega t/2} \right) \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\frac{\alpha}{2} e^{\frac{i\omega t}{2}} \\ \sin\frac{\alpha}{2} e^{-\frac{i\omega t}{2}} \end{pmatrix}$$

$$= \frac{3}{4}\hbar^2 \left(\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2} \right) = \frac{3}{4}\hbar^2$$

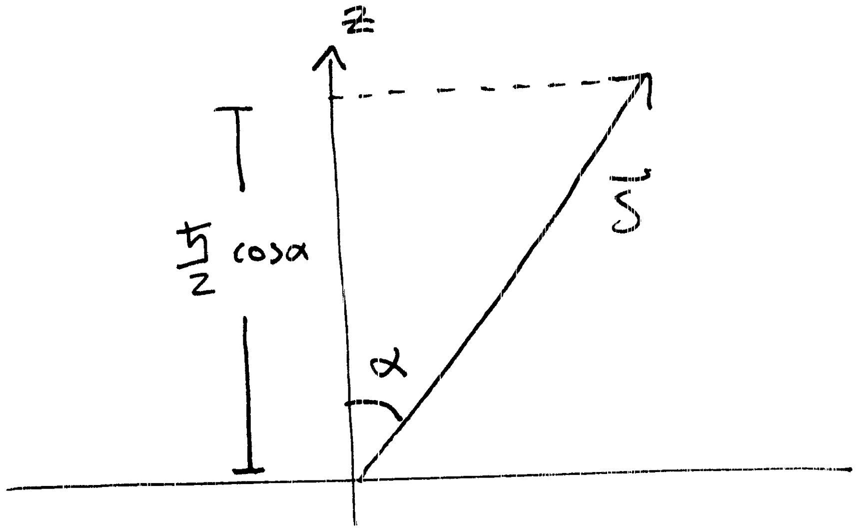
$\Longrightarrow$ The total angular momentum is constant in time.

$$\langle \psi | S_z | \psi \rangle =$$

$$\left(\cos\frac{\alpha}{2} e^{-i\omega t/2}, \sin\frac{\alpha}{2} e^{i\omega t/2} \right) \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\frac{\alpha}{2} e^{i\omega t/2} \\ \sin\frac{\alpha}{2} e^{-i\omega t/2} \end{pmatrix}$$

$$= \left(\cos\frac{\alpha}{2} e^{-\frac{i\omega t}{2}}, \sin\frac{\alpha}{2} e^{\frac{i\omega t}{2}} \right) \frac{\hbar}{2} \begin{pmatrix} \cos\frac{\alpha}{2} e^{i\omega t/2} \\ -\sin\frac{\alpha}{2} e^{-\frac{i\omega t}{2}} \end{pmatrix}$$

$$= \frac{\hbar}{2} \left(\cos^2\frac{\alpha}{2} - \sin^2\alpha/2 \right) = \frac{\hbar}{2} \cos\alpha$$



Compute $\langle \psi | S_x | \psi \rangle$

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\langle \psi | S_x | \psi \rangle =$$

$$\left(\cos\frac{\alpha}{2} e^{-i\omega t/2},\ \sin\frac{\alpha}{2} e^{i\omega t/2}\right) \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\frac{\alpha}{2} e^{i\omega t/2} \\ \sin\frac{\alpha}{2} e^{-i\omega t/2} \end{pmatrix}$$

$$= \frac{\hbar}{2}\left(\cos\frac{\alpha}{2} e^{-i\omega t/2},\ \sin\frac{\alpha}{2} e^{i\omega t/2}\right) \begin{pmatrix} \sin\frac{\alpha}{2} e^{-i\omega t/2} \\ \cos\frac{\alpha}{2} e^{i\omega t/2} \end{pmatrix}$$

$$= \frac{\hbar}{2} \underbrace{\cos\frac{\alpha}{2} \sin\frac{\alpha}{2}}_{\frac{\sin\alpha}{2}} \underbrace{\left(e^{-i\omega t} + e^{i\omega t}\right)}_{2\cos\omega t}$$

$$\langle S_x \rangle = \frac{\hbar}{2} \sin\alpha \cos\omega t$$

The same procedure yields

$$\langle S_y \rangle = -\frac{\hbar}{2} \sin\alpha \sin\omega t$$

$\Longrightarrow$ The x, y component of $\vec{S}$ rotates at a frequency ω