Linear Transformations

A quantum mechanical operator $\hat{A}$ is a linear transformation that maps one vector in the state space $\mathbb{V}$ to another vector in $\mathbb{V}$.

$$|b\rangle = \hat{A}|a\rangle$$

<u>Dfn Linear Transformation</u> - An operator is linear if

$$\hat{A}\left(c_1|a\rangle + c_2|b\rangle\right) = c_1\hat{A}|a\rangle + c_2\hat{A}|b\rangle$$

<u>Operator Inverse</u> $\hat{A}^{-1}$ - The inverse of $\hat{A}$ is an operator $\hat{A}^{-1}$ s.t.

$$\hat{A}^{-1}\hat{A}|a\rangle = |a\rangle$$

for all $|a\rangle$.

<u>Products of Operators</u> The operator $\hat{T}\hat{S}$ is defined as $\hat{T}\hat{S}|a\rangle = \hat{T}\left(\hat{S}|a\rangle\right)$

Inverse of a Product

$$(\hat{T}\hat{S})^{-1} = \hat{S}^{-1}\hat{T}^{-1}$$

Proof

$$(\hat{T}\hat{S})^{-1}\hat{T}\hat{S}|a\rangle = \hat{S}^{-1}\hat{T}^{-1}\hat{T}\hat{S}|a\rangle$$

$$= \hat{S}^{-1}\hat{S}|a\rangle$$

$$= |a\rangle \quad \text{for all } |a\rangle$$

Adjoint of Operator $\hat{T}^{+}$ - The adjoint of $\hat{T}$

is an operator s.t.

$$I(\hat{T}^{+}|a\rangle, |b\rangle) = I(|a\rangle, \hat{T}|b\rangle)$$

for all $|a\rangle, |b\rangle$ where I is the inner product

Dfn Hermitian (Self-Adjoint)

$$\hat{T} = \hat{T}^{+}$$

Dfn Unitary

$$\hat{T}^{+} = \hat{T}^{-1}$$

<u>Ex</u> Prove $(\hat{A}\hat{B})^{+} = \hat{B}^{+}\hat{A}^{+}$

<u>Sln</u>

$$I\left((\hat{A}\hat{B})^{+}|a\rangle, |b\rangle\right) = I\left(|a\rangle, \hat{A}\hat{B}|a\rangle\right)$$

$$= I\left(\hat{A}^{+}|a\rangle, \hat{B}|b\rangle\right)$$

$$= I\left(\hat{B}^{+}\hat{A}^{+}|a\rangle, |b\rangle\right)$$

~~for~~ all $|a\rangle, |b\rangle$ by definition of adjoint.