Matrix Example

Consider a system whose state space is spanned by three vectors $\{|\alpha\rangle, |\beta\rangle, |\gamma\rangle\}$.

The Hamiltonian of the system is

$$\hat{H} = \hbar\omega|\alpha\rangle\langle\alpha| + 2\hbar\omega|\beta\rangle\langle\beta| + 2\hbar\omega|\gamma\rangle\langle\gamma|$$

The Hamiltonian can be represented by a matrix

$$\hat{H} = \begin{pmatrix} \langle\alpha|\hat{H}|\alpha\rangle & \langle\alpha|\hat{H}|\beta\rangle & \langle\alpha|\hat{H}|\gamma\rangle \\ \langle\beta|\hat{H}|\alpha\rangle & \langle\beta|\hat{H}|\beta\rangle & \langle\beta|\hat{H}|\gamma\rangle \\ \langle\gamma|\hat{H}|\alpha\rangle & \langle\gamma|\hat{H}|\beta\rangle & \langle\gamma|\hat{H}|\gamma\rangle \end{pmatrix}$$

$$= \hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

The eigenvalues of the Hamiltonian are

$$\lambda = \hbar\omega, 2\hbar\omega$$

and the eigenvectors are

$$|\hbar\omega\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad |2\hbar\omega, 1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$|2\hbar\omega, 2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$|\hbar\omega\rangle = |\alpha\rangle, \quad |2\hbar\omega, 1\rangle = |\beta\rangle, \quad |2\hbar\omega, 2\rangle = |\gamma\rangle$$

Consider also a second observable, a, represented by the matrix $\hat{A}$.

$$\hat{A} = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

which we have encountered before.

Is $\hat{A}$ an observable?

Yes $\hat{A}^{\dagger} = \hat{A} = (\hat{A}^{T})^{*}$ (Hermitian)

We found the eigenvalues and eigenvectors of $\hat{A}$.

$$\lambda = \pm\hbar, 0$$

$$|+\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \qquad |-\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \qquad |0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\lambda = +\hbar \qquad \lambda = -\hbar \qquad \lambda = 0$$

Assume the system is in state

$$|\psi(0)\rangle = \frac{1}{2}|\alpha\rangle + \frac{1}{\sqrt{2}}|\beta\rangle + \frac{1}{2}|\gamma\rangle$$

at time $t=0$.

What energies could be observed with what probability?

$$P(\hbar\omega) = |\langle\alpha|\psi\rangle|^2 = \frac{1}{4}$$

$$P(2\hbar\omega) = |\langle\beta|\psi\rangle|^2 + |\langle\gamma|\psi\rangle|^2$$

$$= \frac{1}{2} + \frac{1}{4} = 3/4$$

$\sum P_i = 1$ ✓

Suppose we perform a measurement and find the energy of the system is $E = \hbar\omega$, the wave function immediately collapses onto $|\hbar\omega\rangle$

$$|\psi\rangle \xrightarrow{M(\hbar\omega)} |\hbar\omega\rangle = |\alpha\rangle$$

measurement

The measurement projects $|\psi\rangle$ onto the space formed by eigenvectors with eigenvalues $\hbar\omega$. The projection operator for this subspace is

$$\mathbb{P}_{\hbar\omega} = |\alpha\rangle\langle\alpha|$$

$$|\psi\rangle \xrightarrow{M(\hbar\omega)} \mathbb{P}_{\hbar\omega}|\psi\rangle = |\alpha\rangle$$

After the measurement of the energy, a second measurement would observe

$$P(\hbar\omega) = 1 \qquad P(2\hbar\omega) = 1$$

Suppose instead we observed $2\hbar\omega$

<u>Projector</u> $\quad \mathbb{P}_{2\hbar\omega} = |\beta\rangle\langle\beta| + |\gamma\rangle\langle\gamma|$

$$|\psi\rangle \xrightarrow{M(2\hbar\omega)} \mathbb{P}_{2\hbar\omega}|\psi\rangle = \frac{1}{\sqrt{2}}|\beta\rangle + \frac{1}{2}|\gamma\rangle = |\psi'\rangle$$

But we also have to renormalize,

$$\langle\psi'|\psi'\rangle = \frac{1}{2} + \frac{1}{4} = 3/4$$

$$|\psi\rangle \text{ after measurement} = \frac{|\psi'\rangle}{\sqrt{\langle\psi'|\psi'\rangle}}$$

$$= \sqrt{\frac{2}{3}}\,|\beta\rangle + \sqrt{\frac{1}{3}}\,|\gamma\rangle$$

After the measurement,

$$P(2\hbar\omega) = 1 \qquad P(\hbar\omega) = 0$$

Now consider a measurement of $\hat{A}$ at $t=0$.

$$|\psi\rangle = |+\rangle$$

$$P(\hbar) = |\langle +|\psi\rangle|^2 = 1$$

$$P(0) = |\langle 0|\psi\rangle|^2 = 0$$

$$P(-\hbar) = |\langle -|\psi\rangle|^2 = 0$$

since the system is already in a eigenstate of $\hat{A}$.

$$|+\rangle = |\psi\rangle \xrightarrow{M(\hbar)} |\psi\rangle = |+\rangle$$

Now consider two measurements

$$|+\rangle = |\psi\rangle \xrightarrow{M(\hbar\omega)} |\alpha\rangle \xrightarrow{M(\hbar)} |+\rangle$$

when $|\psi\rangle = |\alpha\rangle$

$$P(\hbar) = |\langle +|\psi\rangle|^2 = \frac{1}{4}$$

$$P(0) = |\langle 0|\psi\rangle|^2 = \frac{1}{2}$$

$$P(-\hbar) = |\langle -|\psi\rangle|^2 = \frac{1}{4}$$

$\Rightarrow$ Our measurment of The energy changed the probabilities ~~for~~ for our measurement of $\hat{A}$.

How do we calculate $|\psi(t)\rangle$?

$\Longrightarrow$ Time evolution operator

$$|\psi(t)\rangle = \hat{U}(t, t_0)\,|\psi(t_0)\rangle$$

$$= e^{-\frac{i\hat{H}(t-t_0)}{\hbar}}\,|\psi(t_0)\rangle$$

Represent $\hat{U}$ has a matrix.

$$\underset{\sim}{U} = \begin{pmatrix} \langle\alpha|\hat{U}|\alpha\rangle & \langle\alpha|\hat{U}|\beta\rangle & \langle\alpha|\hat{U}|\gamma\rangle \\ \langle\beta|\hat{U}|\alpha\rangle & \langle\beta|\hat{U}|\beta\rangle & \langle\beta|\hat{U}|\gamma\rangle \\ \langle\gamma|\hat{U}|\alpha\rangle & \langle\gamma|\hat{U}|\beta\rangle & \langle\gamma|\hat{U}|\gamma\rangle \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\omega(t-t_0)} & 0 & 0 \\ 0 & e^{-2i\omega(t-t_0)} & 0 \\ 0 & 0 & e^{-2i\omega(t-t_0)} \end{pmatrix}$$

First calculate time evolution with operators.

$$|\psi(t_0)\rangle = \frac{1}{2}|\alpha\rangle + \frac{1}{\sqrt{2}}|\beta\rangle + \frac{1}{2}|\gamma\rangle$$

$$|\psi(t)\rangle = \hat{U}\,|\Psi(t_0)\rangle =$$

$$\frac{1}{2}e^{-\frac{i\hat{H}(t-t_0)}{\hbar}}|\alpha\rangle + \frac{1}{\sqrt{2}}e^{-\frac{i\hat{H}(t-t_0)}{\hbar}}|\beta\rangle + \frac{1}{2}e^{-\frac{i\hat{H}(t-t_0)}{\hbar}}|\gamma\rangle$$

$$= \frac{1}{2}e^{-i\omega(t-t_0)}|\alpha\rangle + \frac{1}{\sqrt{2}}e^{-2i\omega(t-t_0)}|\beta\rangle + \frac{1}{2}e^{-2i\omega(t-t_0)}|\gamma\rangle$$

Now with matrices,

$$|\psi(t_0)\rangle = \begin{pmatrix} \langle\alpha|\psi(t_0)\rangle \\ \langle\beta|\psi(t_0)\rangle \\ \langle\gamma|\psi(t_0)\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}$$

$$|\psi(t_0)\rangle = \hat{U}\,|\psi(t_0)\rangle$$

$$= \begin{pmatrix} e^{-i\omega(t-t_0)} & 0 & 0 \\ 0 & e^{-2i\omega(t-t_0)} & 0 \\ 0 & 0 & e^{-2i\omega(t-t_0)} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}\, e^{-i\omega(t-t_0)} \\ \frac{1}{\sqrt{2}}\, e^{-2i\omega(t-t_0)} \\ \frac{1}{2}\, e^{-2i\omega(t-t_0)} \end{pmatrix} = \begin{pmatrix} \langle\alpha|\psi(t)\rangle \\ \langle\beta|\psi(t)\rangle \\ \langle\gamma|\psi(t)\rangle \end{pmatrix}$$

Expectation Value of Energy

$$\langle E \rangle = \langle \psi | \hat{H} | \psi \rangle$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \hbar\omega & 0 & 0 \\ 0 & 2\hbar\omega & 0 \\ 0 & 0 & 2\hbar\omega \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2}\hbar\omega \\ \frac{1}{\sqrt{2}} \cdot 2\hbar\omega \\ \frac{1}{2} \cdot 2\hbar\omega \end{pmatrix}$$

$$= \frac{1}{4}\hbar\omega + \frac{1}{2} \cdot 2\hbar\omega + \frac{1}{4} \cdot 2\hbar\omega$$

$$= 1\tfrac{3}{4}\hbar\omega$$

$\Rightarrow$ Not an observable outcome of any single experiment; the average of many experiments

$\Rightarrow$ Same as $\sum c_i^* c_i E_i$

Compute $\langle E \rangle(t) = \langle \psi(t) | \hat{H} | \psi(t) \rangle$

$$= \begin{pmatrix} \frac{1}{2} e^{i\omega t} & \frac{1}{\sqrt{2}} e^{2i\omega t} & \frac{1}{2} e^{2i\omega t} \end{pmatrix} \begin{pmatrix} \hbar\omega & 0 & 0 \\ 0 & 2\hbar\omega & 0 \\ 0 & 0 & 2\hbar\omega \end{pmatrix}$$

$$\cdot \begin{pmatrix} \frac{1}{2} e^{-i\omega t} \\ \frac{1}{\sqrt{2}} e^{-2i\omega t} \\ \frac{1}{2} e^{-2i\omega t} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} e^{i\omega t} & \frac{1}{\sqrt{2}} e^{2i\omega t} & \frac{1}{2} e^{i\omega t} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \hbar\omega e^{-i\omega t} \\ \frac{1}{\sqrt{2}} 2\hbar\omega \, e^{-2i\omega t} \\ \frac{1}{2} 2\hbar\omega \, e^{-2i\omega t} \end{pmatrix}$$

$$= \frac{1}{4} \hbar\omega + \frac{1}{2} 2\hbar\omega + \frac{1}{4} 2\hbar\omega$$

$$= 1\tfrac{3}{4} \hbar\omega \qquad \text{constant}$$

where I have selected $t_0 = 0$.

$\Rightarrow$ Note, we could have let the matrices evolve in time and left the state constant

$$\langle \psi(t) | \hat{H} | \psi(t) \rangle = \langle \psi(t_0) | \hat{U}^{-1} \hat{H} \hat{U} | \psi(t_0) \rangle$$

$$\hat{H}(t) = \hat{U}^{-1} \hat{H} \hat{U} \quad \Rightarrow \quad \text{Heisenberg Picture.}$$

Same calculation with Ehrenfest's Thm

$$\frac{d}{dt}\langle E\rangle = \frac{i}{\hbar}\langle \psi(t)|[\hat{H},\hat{H}]|\psi(t)\rangle$$

$$= 0$$

$$\langle E\rangle(t) = \langle E\rangle(t=0) = 1\tfrac{3}{4}\hbar\omega$$

Now let's calculate the expectation value of "a" as a function of time

$$\langle a\rangle(t) = \langle \psi(t)|\,\hat{A}\,|\psi(t)\rangle$$

$$= \left(\tfrac{1}{2}e^{i\omega t}\ \ \tfrac{1}{\sqrt{2}}e^{2i\omega t}\ \ \tfrac{1}{2}e^{2i\omega t}\right)\cdot\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}\tfrac{1}{2}e^{-i\omega t}\\ \tfrac{1}{\sqrt{2}}e^{-2i\omega t}\\ \tfrac{1}{2}e^{-2i\omega t}\end{pmatrix}$$

$$= \frac{\hbar}{\sqrt{2}}\left(\tfrac{1}{2}e^{i\omega t}\ \ \tfrac{1}{\sqrt{2}}e^{2i\omega t}\ \ \tfrac{1}{2}e^{2i\omega t}\right)\begin{pmatrix}\tfrac{1}{\sqrt{2}}e^{-2i\omega t}\\ \tfrac{1}{2}e^{-i\omega t}+\tfrac{1}{2}e^{-2i\omega t}\\ \tfrac{1}{\sqrt{2}}e^{-2i\omega t}\end{pmatrix}$$

$$\langle a \rangle(t) = \frac{\hbar}{\sqrt{2}}\left(\frac{1}{2\sqrt{2}}e^{-i\omega t} + \frac{1}{2\sqrt{2}}\left(e^{i\omega t}+1\right) + \frac{1}{2\sqrt{2}}\right)$$

$$= \frac{\hbar}{4}\left(e^{i\omega t} + e^{-i\omega t} + 2\right)$$

$$= \frac{\hbar}{4}\left(2\cos\omega t + 2\right)$$

$$= \frac{\hbar}{2}\left(\cos\omega t + 1\right) = \hbar\cos 2\omega t$$

Check At $t=0$, $P(\hbar)=1 \Rightarrow \langle a \rangle(0) = \hbar$

$$\langle a \rangle(0) = \hbar\cos 2\omega\cdot 0 = \hbar \quad ✓$$

What about the probabilities of the individual outcomes of a measurement of "o"?

For example,

$$P(-\hbar) = |\langle - | \psi(t) \rangle|^2$$

$$\langle - | \psi \rangle = \begin{pmatrix} \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} e^{-i\omega t} \\ \frac{1}{\sqrt{2}} e^{-2i\omega t} \\ \frac{1}{2} e^{-2i\omega t} \end{pmatrix}$$

$$= \frac{1}{4} e^{-i\omega t} - \frac{1}{2} e^{-2i\omega t} + \frac{1}{4} e^{-2i\omega t}$$

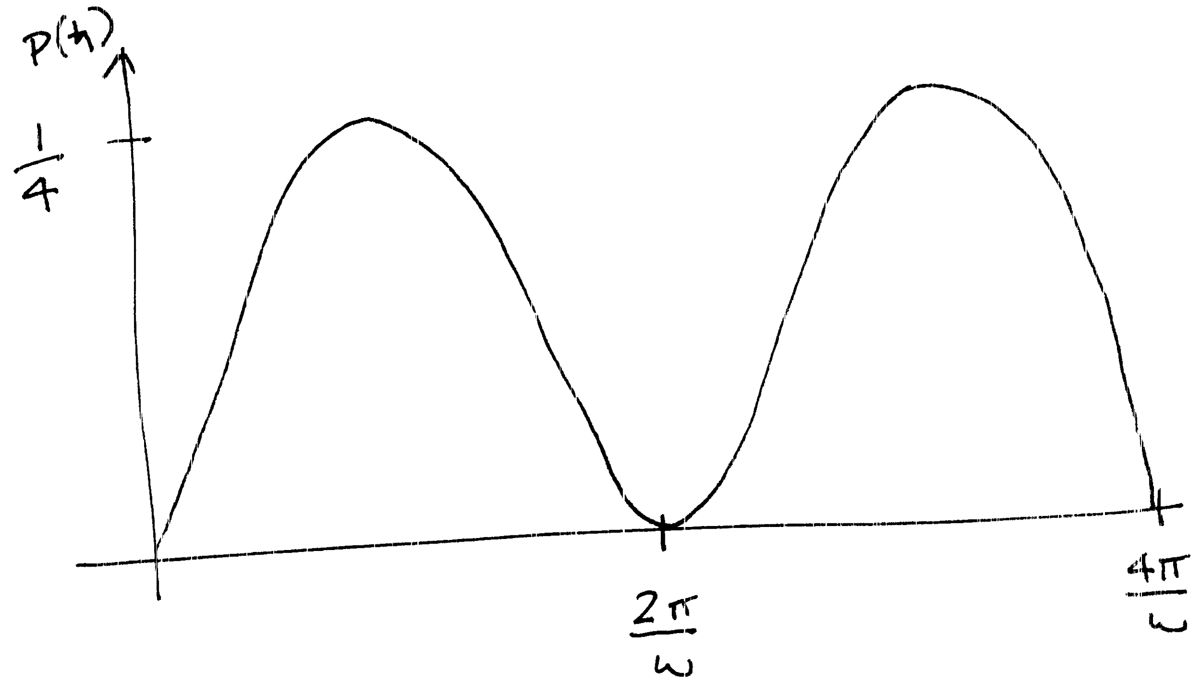
$$= \frac{1}{4} \left(e^{-i\omega t} - e^{-2i\omega t} \right)$$

$$P(-\hbar) = |\langle - | \psi \rangle|^2 = \langle - | \psi \rangle^* \langle - | \psi \rangle$$

$$= \frac{1}{16} \left(e^{i\omega t} - e^{2i\omega t} \right) \left(e^{-i\omega t} - e^{-2i\omega t} \right)$$

$$P(-\hbar) = \frac{1}{16}\left(1 - e^{i\omega t} - e^{-i\omega t} + 1\right)$$

$$= \frac{1}{8}\left(1 - \cos\omega t\right)$$



$\Longrightarrow$ The individual observation probabilities oscillate.

Compute $\frac{d\langle a\rangle}{dt} = \frac{i}{\hbar}\langle\psi|[\hat{H},\hat{A}]|\psi\rangle$

$$\hat{H}\hat{A} = \begin{pmatrix} \hbar\omega & 0 & 0 \\ 0 & 2\hbar\omega & 0 \\ 0 & 0 & 2\hbar\omega \end{pmatrix} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \frac{\hbar^2\omega}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$$

$$\hat{A}\hat{H} = \frac{\hbar^2\omega}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \frac{\hbar^2\omega}{\sqrt{2}} \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}$$

$$[\hat{H},\hat{A}] = \hat{H}\hat{A} - \hat{A}\hat{H} = \frac{\hbar^2\omega}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\langle \psi(t) | [\hat{H}, \hat{A}] | \psi(t) \rangle$$

$$= \frac{\hbar^2 \omega}{\sqrt{2}} \begin{pmatrix} \frac{1}{2} e^{i\omega t} & \frac{1}{\sqrt{2}} e^{2i\omega t} & \frac{1}{2} e^{2i\omega t} \end{pmatrix}$$

$$\cdot \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} e^{-i\omega t} \\ \frac{1}{\sqrt{2}} e^{-2i\omega t} \\ \frac{1}{2} e^{-2i\omega t} \end{pmatrix}$$

$$= \frac{\hbar^2 \omega}{\sqrt{2}} \begin{pmatrix} \frac{1}{2} e^{i\omega t} & \frac{1}{\sqrt{2}} e^{2i\omega t} & \frac{1}{2} e^{2i\omega t} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} e^{-2i\omega t} \\ \frac{1}{2} e^{-i\omega t} \\ 0 \end{pmatrix}$$

$$= \frac{\hbar^2 \omega}{\sqrt{2}} \left(-\frac{1}{2\sqrt{2}} e^{-i\omega t} + \frac{1}{2\sqrt{2}} e^{i\omega t} \right)$$

$$= \frac{\hbar^2 \omega}{4} \underbrace{\left(e^{i\omega t} - e^{-i\omega t} \right)}_{2i \sin \omega t} = \frac{i \hbar^2 \omega}{2} \sin \omega t$$

$$\frac{d}{dt}\langle a\rangle = \frac{i}{\hbar}\langle\psi|[\hat{H},\hat{A}]|\psi\rangle$$

$$= -\frac{\hbar\omega}{2}\sin\omega t$$

$$\langle a\rangle(t) = \frac{\hbar}{2}\cos\omega t + C$$

Uncertainty Relation $\sigma_E^2\sigma_a^2 \geq ?$

$$\sigma_E^2\sigma_a^2 \geq \left(\frac{1}{2i}\langle\psi|[\hat{H},\hat{A}]|\psi\rangle\right)^2$$

$$= \left(\frac{1}{2i}\cdot\frac{i\hbar^2\omega}{2}\sin\omega t\right)^2$$

$$= \frac{\hbar^4\omega^2}{16}\sin^2\omega t$$