Some Additional SHO Points

Results from Problem 3.13

(1) $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$

(2) $[\hat{X}^n, \hat{P}] = i\hbar n \hat{X}^{n-1}$

$\Rightarrow$ $\hat{P}$ takes a derivative with respect to x

(3) $[f(\hat{X}), \hat{P}] = i\hbar \dfrac{df}{dx}$

(4) By analogy, $[\hat{X}, f(\hat{P})] = i\hbar \dfrac{df}{dp}$

$\cancel{[\hat{P}, f(\hat{X})]} =$

Ehrenfest's Thm (classical Mechanics)

$$\hat{H} = \frac{\hat{P}^2}{2m} + V(\hat{x})$$

$$\frac{d}{dt}\langle x \rangle = \frac{i}{\hbar}\langle [\hat{H}, \hat{x}] \rangle + \underbrace{\left\langle \frac{\partial \hat{x}}{\partial t} \right\rangle}_{=\,0}$$

$$[\hat{H}, \hat{x}] = \left[\frac{\hat{P}^2}{2m} + V(\hat{x}), \hat{x}\right]$$

$$= \left[\frac{\hat{P}^2}{2m}, \hat{x}\right] + \underbrace{[V(\hat{x}), \hat{x}]}_{=\,0}$$

$$= -\frac{1}{2m}[\hat{x}, \hat{P}^2] = -\frac{1}{2m} i\hbar \frac{d\hat{P}^2}{d\hat{P}}$$

$$= -\frac{i\hbar}{2m}\hat{P}$$

$$\frac{d}{dt}\langle x \rangle = \frac{i}{\hbar}\left\langle \frac{-i\hbar}{2m}\hat{P} \right\rangle = \frac{\langle \hat{P} \rangle}{m}$$

$$\frac{d\langle p\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H},\hat{p}]\rangle + \underbrace{\left\langle\frac{\partial\hat{p}}{\partial t}\right\rangle}_{=\,0}$$

$$[\hat{H},\hat{p}] = \left[\frac{\hat{p}^2}{2m} + V(\hat{x}),\ \hat{p}\right]$$

$$= [V(\hat{x}),\hat{p}] = i\hbar\,\frac{dV(\hat{x})}{d\hat{x}}$$

3.13 results

$$\frac{d\langle p\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H},\hat{p}]\rangle = \frac{i}{\hbar}\left\langle i\hbar\,\frac{dV}{dx}\right\rangle$$

$$= +\left\langle -\frac{dV}{dx}\right\rangle = \text{force}$$

Back to SHO

I used the Cohen + Tannoudji raising and lowering operators. I want to switch to Griffiths' notation which is a bit less natural, but has some benefits. The proof of the previous lecture carries through unchanged.

Redefine

$$\hat{a}_+ = \hat{a} = \frac{1}{\sqrt{2\hbar m\omega}}\left(-i\hat{P} + m\omega\hat{X}\right)$$

$$\hat{a}_- = \hat{a}^\dagger = \frac{1}{\sqrt{2\hbar m\omega}}\left(i\hat{P} + m\omega\hat{X}\right)$$

which gives

$$\hat{H} = \hbar\omega\left(\hat{a}_+ a_- + \frac{1}{2}\right) = \hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)$$

and $[\hat{a}_-, \hat{a}_+] = 1$

<u>Dfn Number Operator</u> $(\hat{N})$ - If $|\phi_n\rangle$ is an energy eigenstate of the SHO

$$\hat{H}|\phi_n\rangle = E|\phi_n\rangle$$

the $\hat{N}|\phi_n\rangle = n|\phi_n\rangle$

<u>Normalized Ladder Operators</u> If $|\phi_n\rangle$ are the normalized energy eigenstates, then

$$\hat{a}_+|\phi_n\rangle = \sqrt{n+1}\,|\phi_{n+1}\rangle$$

$$\hat{a}_-|\phi_n\rangle = \sqrt{n}\,|\phi_{n-1}\rangle$$ (note works for $n=0$)

When working with the SHO, it is almost always easier to work with $\hat{a}, \hat{a}^{+}$, so

$$\hat{a} + \hat{a}^{+} = \frac{1}{\sqrt{2\hbar m\omega}}\, 2m\omega\, \hat{X}$$

$$\hat{X} = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a} + \hat{a}^{+})$$

and

$$\hat{p} = i \sqrt{\frac{\hbar m \omega}{2}} \left(\hat{a}_{+} - \hat{a}_{-} \right)$$