Multiple Particles

Our work with the addition of angular momentum introduced the concept of multiple-particle systems. In general, each particle has a spin and a location, so the single particle states look like

$$|\psi_i\rangle = |\psi(\vec{r}_i)\rangle \otimes |s_i, m_i\rangle$$

we can then form the full wavefunction as a massive product

$$|\psi\rangle = \sum_{\alpha} c_\alpha \prod_i |\psi(\vec{r}_i)\rangle \otimes |s_i\, m_i\rangle$$

We can represent this as a wavefunction as

$$\psi(\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots \vec{r}_n, t)\ \vec{S}(\vec{s}_1, \vec{s}_2 \ldots \vec{s}_n)$$

where $\vec{S}$ is a vector in the appropriate spin space.

If we focus on only the position parts, that is if we ask only questions that can be answered without specifying the spin we have a classical Hamiltonian

$$H = \sum \frac{p_i^2}{2m} + V(\vec{r}_1, \ldots, \vec{r}_n, t)$$

and a quantum Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla_1^2 - \frac{\hbar^2}{2m}\nabla_2^2 \cdots - \frac{\hbar^2}{2m}\nabla_n^2 + V(\vec{R}_1, \vec{R}_2 \cdots \vec{R}_n, t)$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial z_1^2}$$

The spacial part of the wavefunction must be normalized.

$$1 = \int d\vec{r}_1\, d\vec{r}_2 \cdots d\vec{r}_n \left| \psi(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_n, t) \right|^2$$

so the probability density is

$$\rho = |\psi|^2 = \psi^* \psi$$

We can ask questions like

$$P(x_1 \in [0,a]) = \int_0^a dx_1 \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dz_1 \int d\vec{r}_2 \cdots d\vec{r}_n \, |\psi|^2$$

= probability the x component of particle 1 is between $0, a$ with no constraints on the location of the other particles or the components of $\vec{r}_1$.

$$P(x_1 \in [0,a],\ y_2 \in [0,a])$$

$$= \int_0^a dx_1 \int_{-\infty}^{\infty} dy_1 \int_{-\infty}^{\infty} dz_1 \int_{-\infty}^{\infty} dx_2 \int_0^a dy_2 \int_{-\infty}^{\infty} dz_2 \, |\psi|^2$$

If the particles do not interact, the the potential separates

$$V(\vec{r}_1, r_2 \cdots r_n, t) = \sum_i V_i(\vec{r}_i, t)$$

and the Hamiltonian can be separated into N single particle Hamiltonians

$$H = \sum \frac{p_i^2}{2m_i} + V(\vec{r}_i, t)$$

If V is time independent, then each of these one particle systems can be solve as a TISE.

$$H_i = \frac{p_i^2}{2m_i} + V_i(\vec{r}_i)$$

$$H_i |\psi_i\rangle = E_i |\psi_i\rangle$$

$$-\frac{\hbar^2}{2m_i} \nabla_i^2 \psi_i(\vec{r}_i) + V_i(\vec{r}_i)\psi_i(\vec{r}_i) = E_i \psi_i(\vec{r}_i)$$

and the total energy is

$$E = \sum E_i$$

The $\psi_i(\vec{r}_i)$ notation is cumbersome and is usually abbrevated to $\psi(\vec{r}_i)$, but one should realize $\psi(\vec{r}_i)$ may have no similarity to $\psi(\vec{r}_j)$.

In general, the solutions to any of the i SE's will be quantized with allowed energies

$$\{ E_i^1, E_i^2, \ldots E_i^n \}$$

Each energy will have a different eigenfunction

$$\{ \psi_i^1, \psi_i^2, \psi_i^3 \ldots \psi_i^n \}$$

representing a different state of the particle.

We will label the particle with i, j, k subscripts and the energy states with a, b, c ... superscripts leaving

$$H_i \psi_i^a = E_i^a \psi_i$$

for particle i in energy state E_i^a

This is also ~~written~~ written as

$$\psi_i^a = \psi_a(x_i)$$

letting the subscript on the variable identify the particle.

For non-interacting particles, the full wavefunction is

$$\psi(\vec{r}_1, \vec{r}_2 \cdots \vec{r}_n) = \psi_1^a \psi_2^b \psi_3^c \cdots$$

and the total energy

$$E = E_1^a + E_2^b + E_3^c + \cdots$$

If we confine ourselves to one-dimension and two particles this becomes

$$\psi(x_1, x_2) = \psi_1^a \psi_2^b$$

We can calculate single particle expectation values in the normal way

$$\langle x_1 \rangle = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \; x_1 \left| \psi_1^a(x_1)\, \psi_2^a(x_2) \right|^2$$

$$\Big\| \quad \Big\|$$

or

$$P(x_1 < 0) = \int_{0}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 \; \psi_1^a \psi_1^{a*} \psi_2^b \psi_2^{b*}$$