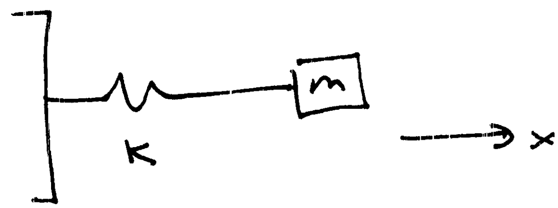
Simple Harmonic Oscillator (SHO)



Consider spring of spring constant k and mass m.

$$F = -kx$$

$$F = ma = m\frac{d^2x}{dt^2} = -kx$$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

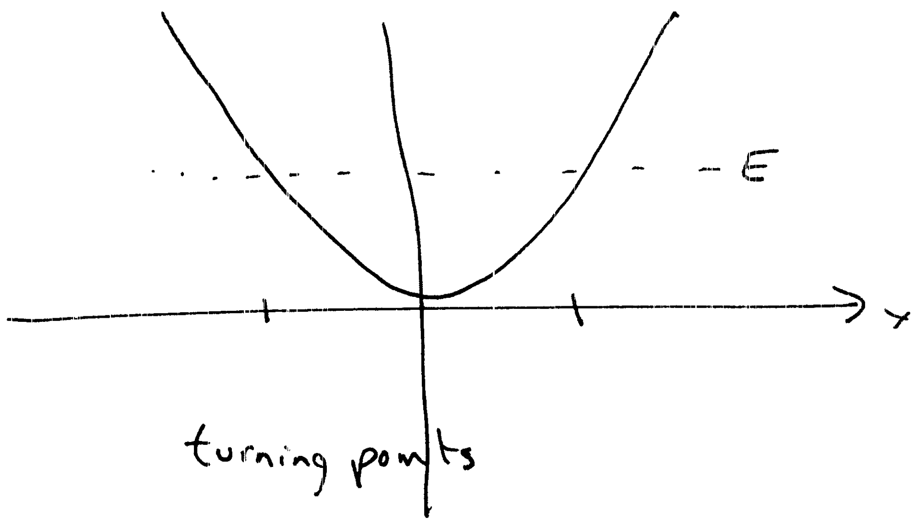
Let $\omega^2 = \frac{k}{m}$

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

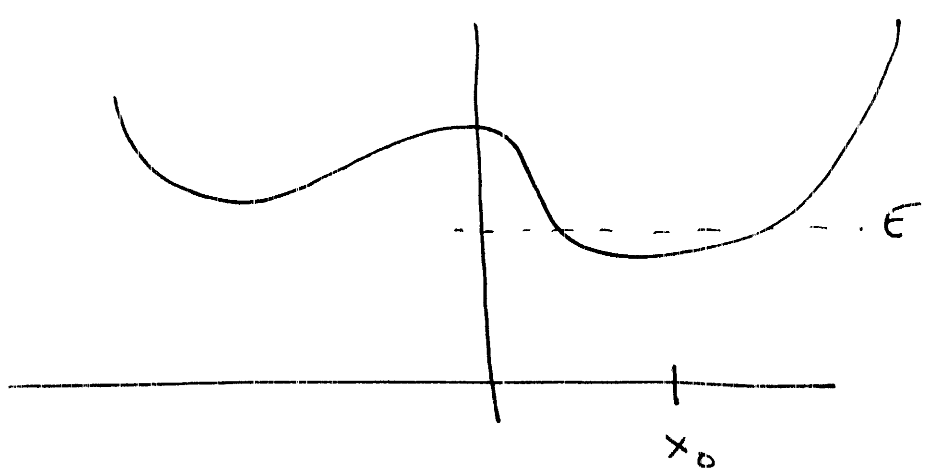
Solutions $x(t) = A\cos\omega t + B\sin\omega t$

Potential Energy

$$V = \frac{1}{2} k x^2$$



The SHO is important because is we take ANY potential and consider motion about a minima



We can approximate the potential with the Taylor expansion

$$V(x) = V(x_0) + \left.\frac{dV}{dx}\right|_{x_0}(x-x_0) + \frac{1}{2}\left.\frac{d^2V}{dx^2}\right|_{x_0}(x-x_0)^2 + \cdots$$

If x_0 is a minima, $\left.\frac{dV}{dx}\right|_{x_0} = 0$ giving

$$V(x) = V(x_0) + \frac{1}{2}\left.\frac{d^2V}{dx^2}\right|_{x_0}(x-x_0)^2$$

Letting $K_{eff} = \left.\frac{d^2V}{dx^2}\right|_{x_0}$ and $x' = x - x_0$ gives

$$V(x) = V(x_0) + \frac{1}{2} k_{eff}\, x'^2$$

so all potentials are SHO near their minima

The total energy of the classical oscillator is

$$E = \frac{1}{2} m v^2 + \frac{1}{2} k x^2$$

and the Hamiltonian with $p = mv$

$$H = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2} k x^2$$

Quantize by replacing classical variables with quantum mechanical operators.

$$\hat{H} = \frac{\hat{P}^2}{2m} + \frac{1}{2} k \hat{X}^2 = \frac{\hat{P}^2}{2m} + \frac{1}{2} m\omega^2 \hat{X}^2$$

Let's use operators to find the energy spectrum

$$\hat{H}|\phi_n\rangle = E_n|\phi_n\rangle$$

<u>Dfn Ladder Operators</u>

$$\hat{a} \equiv \hat{a}_+ = \frac{1}{\sqrt{2m}}\left(\hat{P} + im\omega\hat{X}\right) \quad \text{Raising}$$

$$\hat{a}^\dagger \equiv \hat{a}_- = \frac{1}{\sqrt{2m}}\left(\hat{P} - im\omega\hat{X}\right) \quad \text{Lowering}$$

Write the Hamiltonian in terms of ladder operators.

$$\hat{a}_+\hat{a}_- = \frac{1}{2m}\left(\hat{p} + im\omega\hat{x}\right)\left(\hat{p} - im\omega\hat{x}\right)$$

$$= \frac{1}{2m}\left(\hat{p}^2 + im\omega\left(\hat{x}\hat{p} - \hat{p}\hat{x}\right) + m^2\omega^2\hat{x}^2\right)$$

Recall $[\hat{x}, \hat{p}] = i\hbar = \hat{x}\hat{p} - \hat{p}\hat{x}$

So $$\hat{a}_+\hat{a}_- = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 + \frac{i\omega}{2}\underbrace{[\hat{x}, \hat{p}]}_{i\hbar}$$

$$= \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 - \frac{\hbar\omega}{2}$$

Work on the Hamiltonian, $\omega^2 = k/m$ $\quad k = m\omega^2$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

So $$\hat{a}_+\hat{a}_- = \hat{H} - \frac{\hbar\omega}{2}$$

Therefore,

$$\boxed{\hat{H} = \hat{a}_+ \hat{a}_- + \frac{\hbar\omega}{2}}$$

We will also need $[\hat{a}_-, \hat{a}_+] = \hat{a}_- \hat{a}_+ - \hat{a}_+ \hat{a}_-$

Property of Commutators

$$[A+B, C] = [A, C] + [B, C]$$

$$[\hat{a}_-, \hat{a}_+] = \frac{1}{2m}[\hat{P} - im\omega\hat{X}, \hat{P} + im\omega\hat{X}]$$

$$= \frac{1}{2m}\Big(\underbrace{[\hat{P}, \hat{P}]}_{0} + [\hat{P}, im\omega\hat{X}] + [-im\omega\hat{X}, \hat{P}] + \underbrace{[-im\omega\hat{X}, im\omega\hat{X}]}_{0} \Big)$$

Properties of Commutator

$$[\hat{X}, \hat{X}] = [\hat{P}, \hat{P}] = 0$$

$$[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}]$$

$$[a\hat{A}, \hat{B}] = a[\hat{A}, \hat{B}]$$

$$[\hat{a}_-, \hat{a}_+] = \frac{1}{2m}\left(im\omega \underbrace{[\hat{P}, \hat{X}]}_{-i\hbar} - im\omega \underbrace{[\hat{X}, \hat{P}]}_{i\hbar} \right)$$

$$= \frac{2m\omega\hbar}{2m} = \hbar\omega$$

Now let $\hat{a}_+$ act on on eigenstate (which we don't know) of the Hamiltonian.

$$\hat{H}|\phi_n\rangle = E_n|\phi_n\rangle$$

$$\hat{a}_+|\phi_n\rangle = ?$$

Consider

$$\hat{H}\hat{a}_+|\phi_n\rangle = \left(\hat{a}_+\hat{a}_- + \tfrac{1}{2}\hbar\omega\right)\hat{a}_+|\phi_n\rangle$$

$$= \left(\hat{a}_+\hat{a}_-\hat{a}_+ + \tfrac{1}{2}\hbar\omega\hat{a}_+\right)|\phi_n\rangle$$

$$= \hat{a}_+\left(\hat{a}_-\hat{a}_+ + \tfrac{1}{2}\hbar\omega\right)|\phi_n\rangle$$

Use commutator

$$[\hat{a}_-, \hat{a}_+] = \hat{a}_-\hat{a}_+ - \hat{a}_+\hat{a}_- = \hbar\omega$$

$$\hat{a}_-\hat{a}_+ = \hbar\omega + \hat{a}_+\hat{a}_-$$

$$\hat{H}\hat{a}_+|\phi_n\rangle = \hat{a}_+\Big(\hbar\omega + \underbrace{\hat{a}_+\hat{a}_- + \tfrac{1}{2}\hbar\omega}_{\hat{H}}\Big)|\phi_n\rangle$$

$$= \hat{a}_+(\hat{H} + \hbar\omega)|\phi_n\rangle$$

$$= \hat{a}_+(E_n + \hbar\omega)|\phi_n\rangle$$

$$= (E_n + \hbar\omega)\hat{a}_+|\phi_n\rangle$$

$\Rightarrow$ The state $\hat{a}_+|\phi_n\rangle$ is also an eigenstate of the energy with energy $E_n + \hbar\omega$. The operator $\hat{a}_+$ <u>raised</u> the energy by $\hbar\omega$.

$\Rightarrow$ Note, this does not yield normalized eigenstate.

Through a similar process we can show

$$\hat{a}_-|\phi_n\rangle = (E_n - \hbar\omega)\,\hat{a}_-|\phi_n\rangle$$

so $\hat{a}_-$ lowers the energy by $\hbar\omega$.

The potential is a well, so there should be a lowest energy bound state $|\phi_0\rangle$. Since it is the lowest energy state, it can't be lowered. $\hat{a}_-|\phi_0\rangle = |0\rangle$ (null vector) $= 0$

Try

$$\hat{H}|\phi_0\rangle = \left(\hat{a}_+\hat{a}_- + \tfrac{1}{2}\hbar\omega\right)|\phi_0\rangle$$

$$= \tfrac{1}{2}\hbar\omega|\phi_0\rangle + \hat{a}_+\underbrace{\left(\hat{a}_-|\phi_0\rangle\right)}_{0}$$

$$= \tfrac{1}{2}\hbar\omega|\phi_0\rangle$$

$$\Rightarrow E_0 = \tfrac{1}{2}\hbar\omega$$

$$\Rightarrow E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega$$