SHO Again

The simple harmonic oscillator is

$$\hat{H} = \frac{\hat{P}^2}{2m} + \frac{1}{2} m\omega^2 \hat{X}$$

which has an energy spectrum

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

and eigenvalue eqn

$$\hat{H}\,|\phi_n\rangle = E_n\,|\phi_n\rangle$$

We defined raising and lowering operators

$$\text{Raising} \quad \hat{a}_+ \equiv \hat{a} = \frac{1}{\sqrt{2\hbar m\omega}}\left(-i\hat{P} + m\omega\hat{X}\right)$$

$$\text{Lowering} \quad \hat{a}_- = \hat{a}^+ = \frac{1}{\sqrt{2\hbar m\omega}}\left(i\hat{P} + m\omega\hat{X}\right)$$

The action of these operators move us between states,

$$\hat{a}_+ |\phi_n\rangle = \sqrt{n+1}\, |\phi_{n+1}\rangle$$

$$\hat{a}_- |\phi_n\rangle = \sqrt{n}\, |\phi_{n-1}\rangle$$

where $\hat{a}_- |\phi_0\rangle = 0$.

We can build any energy eigenstate by raising the ground state enough times

$$|\phi_n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}_+)^n |\phi_0\rangle$$

What is $|\phi_0\rangle$ is the $\{|x\rangle\}$ basis?

$$\hat{a}_- |\phi_0\rangle = 0$$

$$\frac{1}{\sqrt{2\hbar m\omega}} \left(i\hat{P} + m\omega \hat{X} \right) |\phi_0\rangle = 0$$

$$\frac{1}{\sqrt{2\hbar m\omega}} \left(i \frac{\hbar}{i} \frac{\partial}{\partial x} + m\omega x \right) \phi_0(x) = 0$$

$$\phi_0(x) = \langle x | \phi_0 \rangle$$

$$\hbar \frac{d}{dx} \phi_0(x) + m\omega x \phi_0(0) = 0$$

$$\phi_0(x) = A e^{-\frac{m\omega}{2\hbar} x^2}$$

Normalize, which everyone has done for this function

$$\phi_0(x) = \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} e^{-\frac{m\omega}{2\hbar} x^2}$$

Therefore,

$$\phi_n = \frac{1}{\sqrt{n!}} (\hat{a}_+)^n |\phi_0\rangle$$

$$= \frac{1}{\sqrt{n!}} \left(\frac{1}{\sqrt{2\hbar m\omega}} \left[-i\left(\frac{\hbar}{i} \frac{d}{dx} \right) + m\omega x \right] \right)^n \phi_0(x)$$

With some work, or the use of Maple, this can be written

$$\phi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi)\, e^{-\xi^2/2}$$

$$\xi \equiv \sqrt{\frac{m\omega}{\hbar}}\, x$$

Hermite Polynomials

$$H_0 = 1$$

$$H_1 = 2x$$

$$H_2 = 4x^2 - 2$$

$$H_3 = 8x^3 - 12x$$

So we may work in operator land or function space land

Expectation Value x in the ground state

$$\langle \phi_0 | \hat{x} | \phi_0 \rangle = \int_{-\infty}^{\infty} \phi_0^* x\, \phi_0 \, dx$$

$$= \sqrt{\frac{m\omega}{\pi\hbar}} \int_{-\infty}^{\infty} x\, e^{-\frac{m\omega}{\hbar} x^2}\, dx = 0$$

or $\quad \hat{x} = \sqrt{\dfrac{\hbar}{2m\omega}}\,(\hat{a}_+ + \hat{a}_-)$

$$\langle \phi_0 | \hat{x} | \phi_0 \rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle \phi_0 | \hat{a}_+ + \hat{a}_- | \phi_0 \rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \Big[\underbrace{\langle \phi_0 | \hat{a}_+ | \phi_0 \rangle}_{= 0 \,=\, \langle \phi_0 | \phi_1 \rangle} + \underbrace{\langle \phi_0 | \hat{a}_- | \phi_0 \rangle}_{= 0} \Big]$$

$$= 0$$

<u>what about $\langle \hat{x}^2 \rangle$</u>?

$$\langle \hat{x}^2 \rangle = \int_{-\infty}^{\infty} x^2 \phi_0^*(x)\, \phi_0(x)\, dx$$

$$= \sqrt{\frac{m\omega}{\pi\hbar}} \int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2}\, dx$$

$$= \langle \phi_0 | \hat{x}^2 | \phi_0 \rangle$$

$$= \frac{\hbar}{2m\omega} \langle \phi_0 | (\hat{a}_+ + \hat{a}_-)^2 | \phi_0 \rangle$$

$$\langle \hat{x}^2 \rangle = \frac{\hbar}{2m\omega} \langle \phi_0 | \hat{a}_+^2 + \hat{a}_-^2 + \hat{a}_+ \hat{a}_- + \hat{a}_- \hat{a}_+ | \phi_0 \rangle$$

$$= \frac{\hbar}{2m\omega} \Big[\langle \phi_0 | \hat{a}_+^2 | \phi_0 \rangle + \langle \phi_0 | \hat{a}_-^2 | \phi_0 \rangle$$

$$+ \langle \phi_0 | \hat{a}_+ \hat{a}_- | \phi_0 \rangle + \langle \phi_0 | \hat{a}_- \hat{a}_+ | \phi_0 \rangle \Big]$$

$$= \frac{\hbar}{2m\omega} \langle \phi_0 | \hat{a}_- \hat{a}_+ | \phi_0 \rangle$$

because

$$\langle \phi_0 | \hat{a}_+^2 | \phi_0 \rangle = \langle \phi_0 | \hat{a}_+ (\hat{a}_+ | \phi_0 \rangle)$$

$$= \langle \phi_0 | \hat{a}_+ | \phi_1 \rangle = \sqrt{2} \langle \phi_0 | \phi_2 \rangle = 0$$

$$\langle \phi_0 | \hat{a}_-^2 | \phi_0 \rangle = \langle \phi_0 | \hat{a}_- (\hat{a}_- | \phi_0 \rangle) = 0$$

$$\langle \phi_0 | \hat{a}_+ (\hat{a}_- | \phi_0 \rangle) = 0$$

$$\hat{a}_+|\phi_0\rangle = \sqrt{0+1}\,|\phi_1\rangle = |\phi_1\rangle$$

$$\hat{a}_-|\phi_1\rangle = \sqrt{1}\,|\phi_0\rangle = |\phi_0\rangle$$

$\cancel{\langle\phi_0|\hat{a}_-\hat{a}_+:}$ $\langle\phi_0|\hat{a}_-\hat{a}_+|\phi_0\rangle = \langle\phi_0|\hat{a}_-|\phi_1\rangle$

$$= \langle\phi_0|\phi_0\rangle = 1$$

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}$$

3.34 The ground state has energy $\frac{1}{2}\hbar\omega$ and the first excited state $3/2\,\hbar\omega$. At $t=0$,

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\theta}|1\rangle\right)$$

since we know the probabilities are equal but we don't know the phase relationship.

At time t, $\quad |\psi\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\omega_0 t}|0\rangle + e^{-i3\omega_0 t + i\theta}|1\rangle\right)$

where I have introduced $\omega_0 = \frac{1}{2}\omega$

$$|\psi\rangle = \frac{e^{-i\omega_0 t}}{\sqrt{2}}\left(|0\rangle + e^{-2i\omega_0 t + i\theta}|1\rangle\right)$$

From chapter 2

$$p = i\sqrt{\frac{\hbar m\omega}{2}}\,(a_+ - a_-) \qquad 2.69$$

$$a_+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$a_-|n\rangle = \sqrt{n}\,|n-1\rangle \qquad 2.66$$

Let $\quad A = \dfrac{e^{-i\omega_0 t}}{\sqrt{2}} \qquad \gamma = e^{-2i\omega_0 t + i\theta}$

$$|\psi\rangle = A\left(|0\rangle + \gamma|1\rangle\right)$$

$$p|\psi\rangle = \left(Ai\sqrt{\frac{\hbar m\omega}{2}}\right)\left(a_+|0\rangle + \gamma a_+|1\rangle - \gamma a_-|1\rangle\right)$$

$$= \left(Ai\sqrt{\frac{\hbar m\omega}{2}}\right)\left(|1\rangle + \gamma\sqrt{2}|2\rangle - \gamma|0\rangle\right)$$

$$\langle\psi|p|\psi\rangle = \left(A^* A i\sqrt{\frac{\hbar m\omega}{2}}\right)\left(\langle 0| + \gamma^*\langle 1|\right)\left(|1\rangle + \gamma\sqrt{2}|2\rangle - \gamma|0\rangle\right)$$

$$= \left(\frac{i}{2}\sqrt{\frac{\hbar m\omega}{2}}\right)\left(\gamma^* - \gamma\right)$$

$$\gamma^* - \gamma = +2i\sin\left(2\omega_0 t - \theta\right)$$

$$\langle\psi|p|\psi\rangle = -\sqrt{\frac{\hbar m\omega}{2}}\sin\left(2\omega_0 t - \theta\right)$$

$$= -\sqrt{\frac{\hbar m\omega}{2}}\sin\left(\omega t - \theta\right)$$

$\langle p \rangle$'s maximum value is $\sqrt{\frac{\hbar m \omega}{2}}$

If this is maximum at $t=0$, $\theta = +\frac{\pi}{2}$

The $-$ sign out front means $\langle p \rangle$ maximum when sin minimum.

$$|\psi\rangle = \frac{e^{-i\omega t/2}}{\sqrt{2}} \left(|0\rangle + i|1\rangle \right)$$

(2.41)

$$\phi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\xi^2/2} = B e^{-\xi^2/2}$$

$$\phi_1 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{2\xi}{\sqrt{2}} e^{-\xi^2/2} = B\sqrt{2}\,\xi\, e^{-\xi^2/2}$$

$$\phi_2 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{4\xi^2-2}{2\sqrt{2}} e^{-\xi^2/2}$$

from 2.85

$$= B\left(\sqrt{2}\,\xi^2 - \frac{1}{\sqrt{2}}\right)e^{-\xi^2/2}$$

$$\xi = \sqrt{\frac{m\omega}{\hbar}}\, x \qquad B \equiv \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

The given wave function is

$$\Psi(x,0) = A(1-2\xi)^2 e^{-\xi^2/2}$$

$$= A(1-4\xi+4\xi^2)\, e^{-\xi^2/2}$$

$$= c_0\phi_0 + c_1\phi_1 + c_2\phi_2$$

$$\Longrightarrow 4A = c_2 B\sqrt{2}$$

Equate powers of ξ

$$\boxed{c_2 = 2\sqrt{2}\,\frac{A}{B}}$$

$$-4A = B\sqrt{2}\, c_1$$

$$\boxed{c_1 = -2\sqrt{2}\,\frac{A}{B}}$$

ξ^0: $$A = c_0 B - c_2 \frac{B}{\sqrt{2}}$$

$$= c_0 B - 2\sqrt{2}\,\frac{A}{B}\left(\frac{B}{\sqrt{2}}\right)$$

$$= c_0 B - 2A$$

$$\boxed{c_0 = 3\,\frac{A}{B}}$$

<u>Normalize to find A</u>

$$c_0^* c_0 + c_1^* c_1 + c_2^* c_2 = 1$$

$$\frac{9A^2}{B^2} + \frac{8A^2}{B^2} + \frac{8A^2}{B^2} = 1$$

$$A = \frac{B}{\sqrt{25}} = \frac{B}{5}$$

$$c_0 = \frac{3}{5} \qquad c_1 = -\frac{2\sqrt{2}}{5} \qquad c_2 = \frac{2\sqrt{2}}{5}$$

$$\psi = \frac{3}{5}\phi_0 - \frac{2\sqrt{2}}{5}\phi_1 + \frac{2\sqrt{2}}{5}\phi_2$$

$$\langle E \rangle = c_0^* c_0 \left(\tfrac{1}{2}\hbar\omega\right) + c_1^* c_1 \left(\tfrac{3}{2}\hbar\omega\right) + c_2^* c_2 \left(\tfrac{5}{2}\hbar\omega\right)$$

$$= \frac{9}{25}\left(\tfrac{1}{2}\hbar\omega\right) + \frac{8}{25}\left(\tfrac{3}{2}\hbar\omega\right) + \frac{8}{25}\left(\tfrac{5}{2}\hbar\omega\right)$$

$$= \frac{\hbar\omega}{50}\left(9 + 24 + 40\right)$$

$$\boxed{\langle E \rangle = \frac{73}{50}\hbar\omega}$$

(b) $$\psi(x,t) = \frac{3}{5}e^{-i\omega t/2}\phi_0 - \frac{2\sqrt{2}}{5}e^{-3i\omega t/2}\phi_1 + \frac{2\sqrt{2}}{5}e^{-5i\omega t/2}\phi_2$$

~~The change affects only the ϕ_1 term changing its sign~~

$$\cancel{\frac{3\omega t}{2}}$$

Factor out the overall phase

$$\psi(x,t) = e^{-i\omega t/2}\left[\frac{3}{5}\phi_0 - \frac{2\sqrt{2}}{5}e^{-i\omega t}\phi_1 + \frac{2\sqrt{2}}{5}e^{-2i\omega T}\phi_2\right]$$

We want to change the sign of the second term without affecting the third term

$$\omega T = \pi$$

$$T = \frac{\pi}{\omega}$$