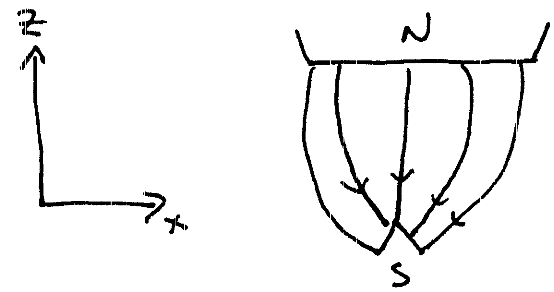
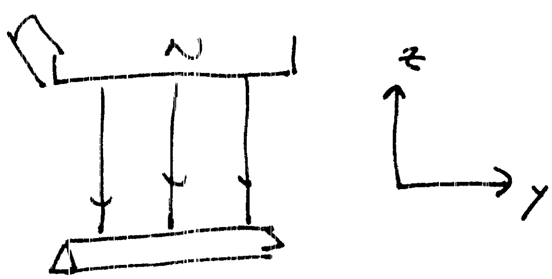
Stern - Gerlach

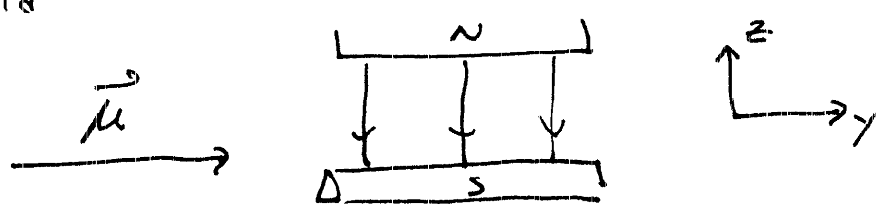
If we send a particle with magnetic moment $\vec{\mu}$ through an anisotropic magnetic field $\vec{B}$, it will experience a force $\vec{F} = \nabla(\vec{\mu}\cdot\vec{B})$.

We will use a field that is on average in the $\hat{z}$ direction. No Monopoles, $\nabla\cdot\vec{B}=0$, forces us to have a field that changes in two directions. Construct a field that changes in the x and z directions, $\vec{B}(x,z)$, as below





Now shoot a ~~part~~ particle with moment $\vec{\mu}$ through the field



Expand $\vec{B}(x,z)$ about its average value

$$\vec{B}_{ave} = B_0 \hat{z}$$

$$\vec{B}(x,z) \approx \alpha x \,\hat{x} + (B_0 + \beta z)\,\hat{z}$$

where α, β are expansion coefficients.

<u>Maxwell</u> $\quad \nabla \cdot \vec{B} = 0 \implies \alpha + \beta = 0$

$$\vec{B}(x,z) = -\alpha x \,\hat{x} + (B_0 + \alpha z)\,\hat{z}$$

where I've redefined alpha to give a positive z direction.

$$\vec{\mu} = \gamma \vec{S}$$

spin matrix

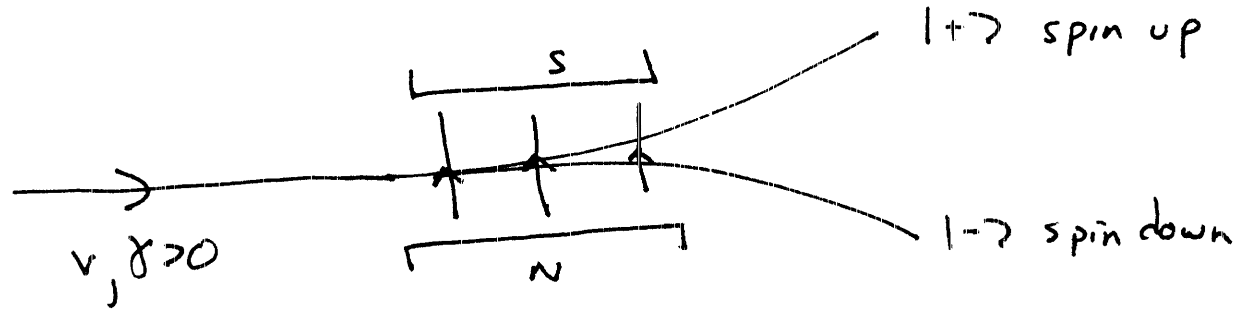
gyromagnetic ratio

$$\vec{\mu} \cdot \vec{B} = -\alpha \gamma S_x x + \gamma (B_0 + \alpha z) S_z$$

$$\vec{F} = \nabla(\vec{\mu} \cdot \vec{B}) = -\alpha \gamma S_x \,\hat{x} + \alpha \gamma S_z \,\hat{z}$$

1. Net force in z-direction.

2. The spin precesses about the z axis, so the x-component averages to zero.

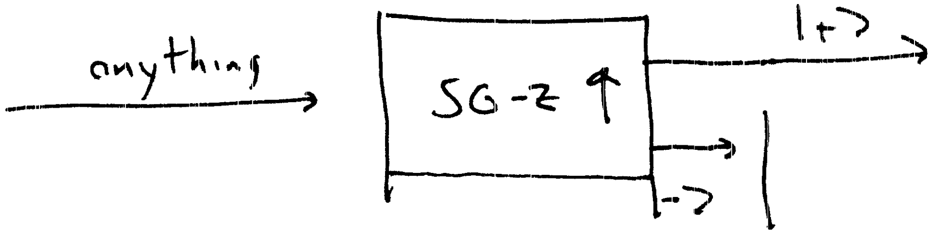


$\Longrightarrow$ Stern-Gerlach apparatus splits beam into two streams based on spin.

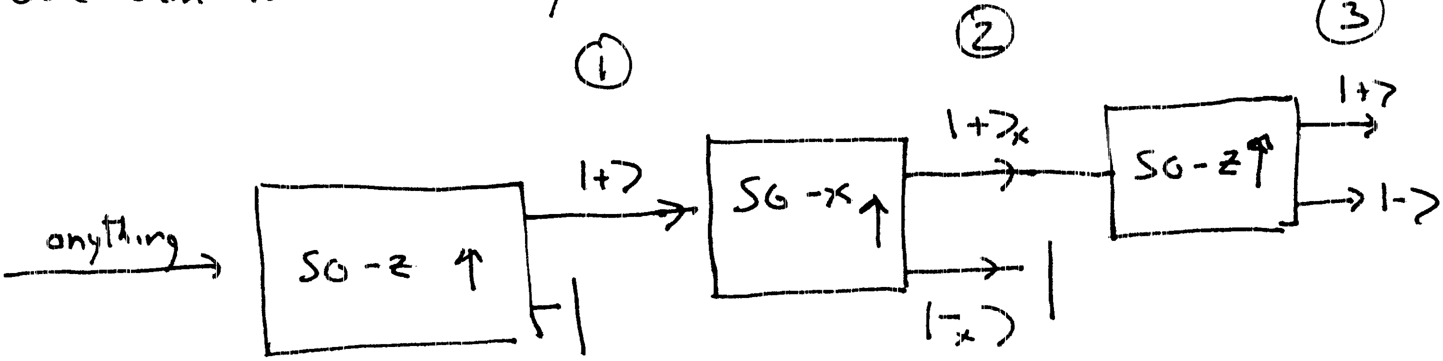
$\Longrightarrow$ Represent a Stern-Gerlach apparatus with z-directed field as

$$\boxed{\text{SG} - z}$$

With an SG apparatus, we can build a pure state by blocking one of the outputs



We can look at sequences of measurements



Analysis

1. A pure $|+\rangle = |+\rangle_z$ state exits the first device

2. The second device applies a non-uniform field is the x direction and splits the beam into $|+\rangle_x$, $|-\rangle_x$ defined by

$$S_x |+\rangle_x = \frac{\hbar}{2} |+\rangle_x \qquad S_x |-\rangle_x = -\frac{\hbar}{2} |-\rangle_x$$

Given a $|+\rangle$ input what are the odds of getting a $|+\rangle_x$ output. $\Longrightarrow$ Expand $|+\rangle$ in the eigenvectors of S_x.

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\det\left(S_x - \lambda \hat{1}\right) = \begin{vmatrix} -\lambda & \hbar/2 \\ \hbar/2 & -\lambda \end{vmatrix} = \lambda^2 - \frac{\hbar^2}{4} = 0$$

$\lambda = \pm \hbar/2$ (which we already know)

$\underline{|+\rangle_x}$

$$\begin{pmatrix} -\hbar/2 & \hbar/2 \\ \hbar/2 & -\hbar/2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

$$a_1 - a_2 = 0$$

$$|+\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$$

Likewise

$$|-\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle$$

Build $|+\rangle$

$$|+\rangle_x + |-\rangle_x = \frac{2}{\sqrt{2}}|+\rangle$$

$$|+\rangle = \underset{\overset{\shortparallel}{c_+}}{\frac{1}{\sqrt{2}}}|+\rangle_x + \underset{\overset{\shortparallel}{c_-}}{\frac{1}{\sqrt{2}}}|-\rangle_x$$

$$P(|+\rangle_x) = c_+^* c_+ = \frac{1}{2}$$

$$P(|-\rangle_x) = c_-^* c_- = 1/2$$

(3) Now run the $|+\rangle_x$ beam through the second SG-z apparatus.

$$|+\rangle_x = \underset{\overset{\shortparallel}{c_+}}{\frac{1}{\sqrt{2}}}|+\rangle + \underset{\overset{\shortparallel}{c_-}}{\frac{1}{\sqrt{2}}}|-\rangle$$

$P(|+\rangle) = 1/2 \qquad P(|-\rangle) = 1/2$

$\Rightarrow$ Note the beam now contains both spin up and spin down, even though all spin down were blocked by first apparatus.