PHYS 4073 - Quantum Mechanics- Test 1

All problems are worth 25 points. Turn in solutions to four of the six problems to be graded. If you turn in more than four solutions, I will grade the first four. You are allowed to drop one-half of a test, so I will take the first two problems turned in as the first half-test and the second two problems turned in as the second half-test. Had a bit of a writer's block on this one.

1 An infinite square well has potential $V = 0$ for $0 < x < a$ and $V = \infty$ otherwise. Consider the energy state immediately above the ground state, the first excited state. Compute the energy of this state for an electron if the well is 10nm wide. Write the normalized wave function of the the state. Sketch the state. Compute the probability of finding the electron within $a/4$ of the right side of the well.

2 Consider the time-dependent wave function $\psi(x,t) = Ae^{ikx}e^{-i\omega t}$. For what potential does this wave function solve the Schrodinger equation if the energy of the system is E? Compute the average momentum of the state and the uncertainty in the momentum. Assume (incorrectly) that ψ is normalized.

3 Consider a finite well with a strong, but not infinite defect, in the center. The potential is

$$V(x) = \begin{cases} 0 & \text{if } x < -a \\ -V_0 & \text{if } -a < x < 0 \\ V_0 & \text{if } 0 < x < a/4 \\ -V_0 & \text{if } a/4 < x < a \\ 0 & \text{if } x > a \end{cases}$$

Sketch the potential. Write the wave function that solves the Schrodinger equation for this potential for the two cases $0 < E < V_0$ and $-V_0 < E < 0$. Report the equations you get by applying the appropriate boundary conditions for $-V_0 < E < 0$. Do not solve the system.

4 A system with an infinite square well potential has the unlikely wave function $\phi(x,0) = Ax^2$. The well extends from 0 to a. Compute the probability of finding the particle in the ground state and the first excited state. Compute the uncertainty in x, σ_x.

5 A particle is directed at a finite step potential where

$$V(x) = \begin{cases} 0 & \text{if } x < 0 \\ V_0 & \text{if } 0 < x \end{cases}$$

The particle has energy $E = 4V_0$. Compute the probability of reflection. What is the reflection probability if $E = V_0/4$?

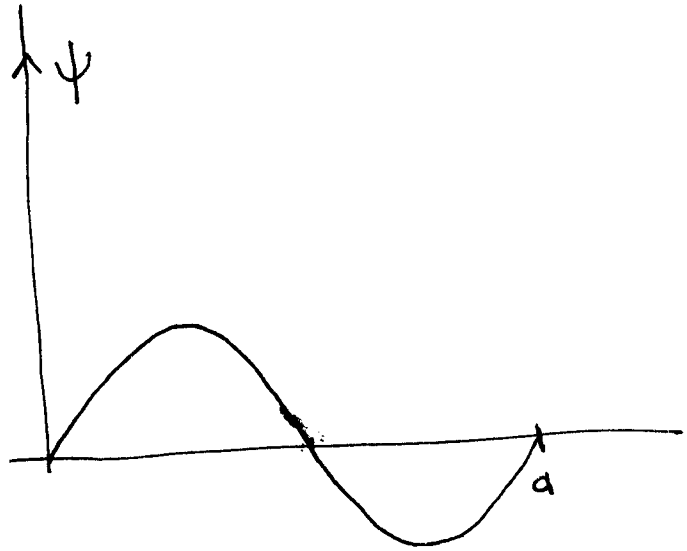
6 A particle is placed in an infinite square well extending from $0 < x < a$. The particle is in the state

$$\psi(x,0) = A\left(\sin^3\left(\frac{2\pi x}{a}\right) + 2\sin\left(\frac{3\pi x}{a}\right)\right)$$

What energies can be observed with what probability? Compute the time dependent wave function $\psi(x,t)$. Don't ignore the $\sin^3$.

1

$$\phi_2 = \sqrt{\frac{2}{a}} \sin \frac{2\pi x}{a}$$



$$E_2 = \frac{\hbar^2 k_2^2}{2m}$$

$$= \frac{\hbar^2}{2m}\left(\frac{2\pi}{a}\right)^2 = \frac{2\hbar^2\pi^2}{ma^2}$$

$$= 2.4\times10^{-21}\,\text{J} = 0.015\,\text{eV}$$

$$P\left(x > \frac{3a}{4}\right) = \int_{\frac{3a}{4}}^{a} \phi_2^* \phi_2 \, dx$$

$$= \frac{2}{a}\int_{\frac{3a}{4}}^{a} \sin^2 \frac{2\pi x}{a} \, dx = \frac{1}{4}$$

Maple

② SE

$$-\frac{\hbar^2}{2m}\frac{d^2\phi}{dx^2} + V(x)\phi = E\phi$$

$$-\frac{\hbar^2}{2m}\frac{d^2\phi}{dx^2} = -\frac{\hbar^2}{2m}(ik)^2 A e^{ikx} e^{-i\omega t}$$

$$= \frac{\hbar^2 k^2}{2m}\psi(x,t)$$

$$\frac{\hbar^2 k^2}{2m}\psi + V\psi = E\psi$$

$$\boxed{V(x) = E - \frac{\hbar^2 k^2}{2m}} \qquad \boxed{V = \hbar\omega - \frac{\hbar^2 k^2}{2m}}$$

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi^* \left(+\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\psi \, dx$$

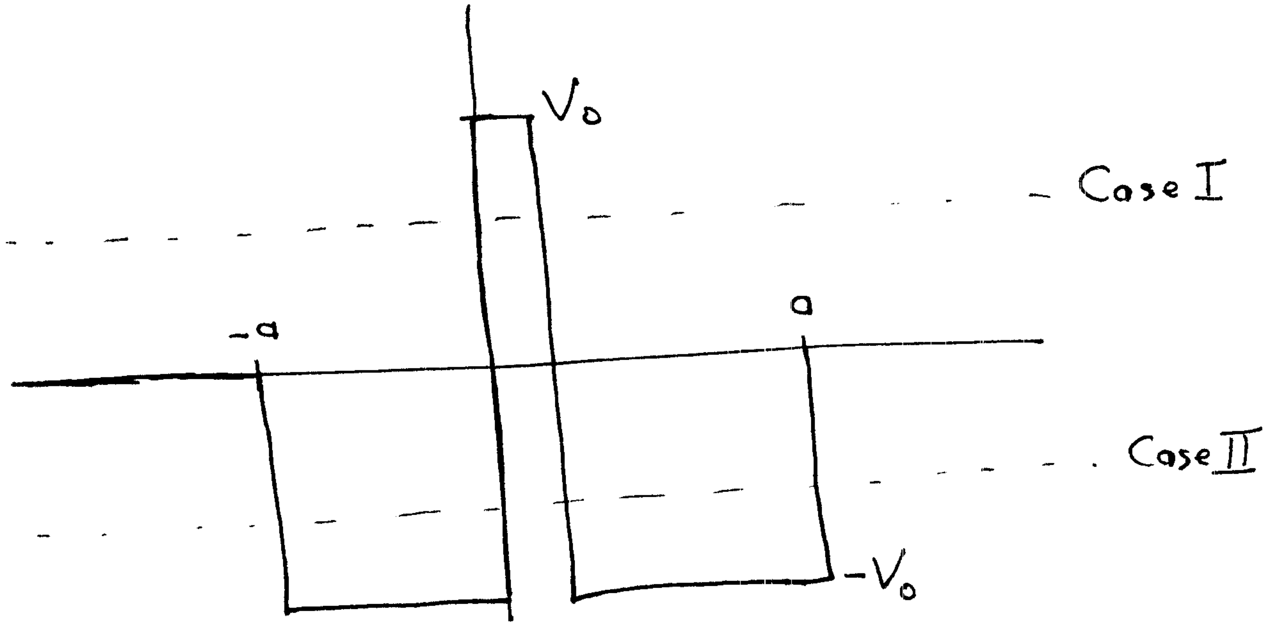
$$= \frac{ik\hbar}{i}\int_{-\infty}^{\infty} \psi^*\psi \, dx = \hbar k$$

$\langle p^2 \rangle = (k\hbar)^2$ by similar methods.

$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = 0$$

$\Longrightarrow$ A state of well-defined momentum.

③



<u>Case I</u>

Let $\quad K = \sqrt{\dfrac{2mE}{\hbar^2}} \qquad l = \sqrt{\dfrac{2m(E+V_0)}{\hbar^2}}$

$$\rho = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$$

$$\psi(x,0) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x < -a \\ Ce^{ilx} + De^{-ikx} & -a < x < 0 \\ Ee^{\rho x} + Fe^{-\rho x} & 0 < x < a/4 \\ Ge^{ilx} + He^{-ilx} & a/4 < x < a \\ Ie^{ikx} & x > a \end{cases}$$

Case II $V_0 < E < 0$

$$\rho = \sqrt{\frac{-2mE}{\hbar^2}} \qquad = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}} = \ell$$

$$k = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}$$

$$\psi(x,0) = \begin{cases} A e^{\rho x} & x < -a \\ B e^{ikx} + C e^{-ikx} & -a < x < 0 \\ D e^{\ell x} + E e^{-\ell x} & 0 < x < a/4 \\ F e^{ikx} + G e^{-ikx} & a/4 < x < a \\ H e^{-\rho x} & x > a \end{cases}$$

Continuity

$x = -a$ $\quad A e^{-\rho a} = B e^{-ika} + C e^{ika}$

$x = 0$ $\quad B + C = D + E$

$x = a/4$ $\quad D e^{\ell a/4} + E e^{-\ell a/4} = F e^{ika/4} + G e^{-ika/4}$

$x = a$ $\quad F e^{ika} + G e^{-ika} = H e^{-\rho a}$

Slope Continuous

$x=-a$: $$A\beta e^{-\beta a} = ikBe^{-ika} - ikCe^{+ika}$$

$x=0$: $$iBk - ikC = D\ell - E\ell$$

$x=a/4$: $$D\ell e^{\ell a/4} - E\ell e^{-\ell a/4} = ikFe^{+ika/4} - ikGe^{-ika/4}$$

$x=a$: $$ikFe^{ika} - ikGe^{-ika} = -\beta He^{-\beta a}$$

(4) Normolize

$$\psi(x,0) = A x^2$$

$$1 = A^2 \int_0^a x^4 \, dx = \frac{A^2 x^5}{5} \Big|_0^a = \frac{A^2 a^5}{5}$$

$$\sqrt{\frac{5}{a^5}} = A$$

$$\langle x \rangle = \int x \, \psi^* \psi \, dx = \frac{5}{a^5} \int_0^a x^5 \, dx$$

$$= \frac{5}{6a^5} x^6 \Big|_0^a = \frac{5}{6} a$$

$$\langle x^2 \rangle = \int_0^a x^2 \psi^* \psi \, dx = \frac{5}{a^5} \int_0^a x^6 \, dx$$

$$= \frac{5x^7}{7a^5} \Big|_0^a = \frac{5}{7} a^2$$

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = a \sqrt{\frac{5}{7} - \left(\frac{5}{6}\right)^2}$$

$$= \cancel{0.37a} \quad 0.14a$$

Ground State

$$\phi_1 = \sqrt{\frac{2}{a}} \sin \frac{\pi x}{a}$$

$$c_1 = \int_0^a \phi_1^* \psi \, dx = \left(\sqrt{\frac{2}{a}}\right)\left(\sqrt{\frac{5}{a^5}}\right) \int_0^a x^2 \sin \frac{\pi x}{a} \, dx$$

$$= \frac{\sqrt{10} \, (\pi^2 - 4)}{\pi^3} \qquad \text{Maple}$$

First Excited State

$$\phi_2 = \sqrt{\frac{2}{a}} \sin \frac{2\pi x}{a}$$

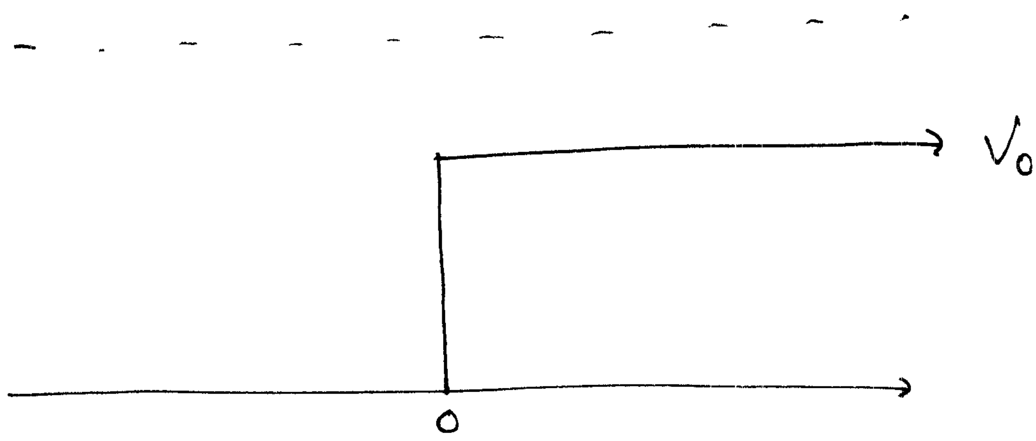
$$c_2 = \sqrt{\frac{2}{a}} \sqrt{\frac{5}{a^5}} \int_0^a x^2 \sin \frac{2\pi x}{a} \, dx$$

$$= -\frac{1}{\pi} \sqrt{\frac{5}{2}} \qquad \text{Maple}$$

$$P_1 = c_1^* c_1 = 0.35$$

$$P_2 = c_2^* c_2 = 0.25$$

(5)



$R = 1$ if $V_0 < E$

Let $k = \sqrt{\frac{2mE}{\hbar^2}}$ $\qquad l = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$

$$\psi = \begin{cases} Ae^{ikx} + Be^{-ikx} & x<0 \\ Ce^{ilx} & x>0 \end{cases}$$

<u>Continuity</u> $\quad A + B = C \qquad (1)$

<u>Slope Continuous</u> $\quad ikA - ikB = ilC \qquad (2)$

Sub ① → ②

$$kA - kB = lA + lB$$

$$(k - l)A = (k + l)B$$

$$R = \left(\frac{B}{A}\right)^2 = \left(\frac{k-l}{k+l}\right)^2 = \left(\frac{\sqrt{E} - \sqrt{E - V_0}}{\sqrt{E} + \sqrt{E - V_0}}\right)^2$$

$$= \left(\frac{2 - \sqrt{3}}{2 + \sqrt{3}}\right)^2 = \text{[illegible]}$$

$$= 5.1 \times 10^{-3}$$

6 Work on $\sin^2 x$

$$\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$$

$$\sin^3\left(\frac{2\pi x}{a}\right) = \frac{3}{4}\sin\left(\frac{2\pi x}{a}\right) - \frac{1}{4}\sin\frac{6\pi x}{a}$$

$$\psi(x,0) = A\left(\frac{3}{4}\sin\frac{2\pi x}{a} + 2\sin\frac{3\pi x}{a} - \frac{1}{4}\sin\frac{6\pi x}{a}\right)$$

$$= A'\left(3\phi_2 + 8\phi_3 - 1\phi_6\right)$$

Normalize

$$1 = \int \psi^* \psi \, dx = A'^2\left(9 + 64 + 1\right)$$

$$A' = \frac{1}{\sqrt{74}}$$

$$\psi(x,0) = \frac{3}{\sqrt{74}}\phi_2 + \frac{8}{\sqrt{74}}\phi_3 - \frac{1}{\sqrt{74}}\phi_6$$

Energy $\quad E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2 \pi^2}{2ma^2} n^2$

$$P(E_2) = \left(\frac{3}{\sqrt{74}}\right)^2 = \frac{9}{74} = 12\% \qquad E_2 = 4\left(\frac{\hbar^2\pi^2}{2ma^2}\right) = \hbar\omega_2$$

$$P(E_3) = \left(\frac{8}{\sqrt{74}}\right)^2 = \frac{64}{74} = 86\% \qquad E_3 = 9\left(\frac{\hbar^2\pi^2}{2ma^2}\right) = \hbar\omega_3$$

$$P(E_6) = \left(\frac{1}{\sqrt{74}}\right)^2 = \frac{1}{74} = 1.4\% \qquad E_6 = 36\left(\frac{\hbar^2\pi^2}{2ma^2}\right) = \hbar\omega_6$$

$$\psi(x,t) = \frac{3}{\sqrt{74}}\phi_2 e^{-i\omega_2 t} + \frac{8}{\sqrt{74}}\phi_3 e^{-i\omega_3 t}$$

$$- \frac{1}{\sqrt{74}}\phi_6 e^{-i\omega_6 t}$$

$$\phi_n = \sqrt{\frac{2}{a}} \sin \frac{n\pi}{a} x$$