# PHYS 4073 - Quantum Mechanics- Test 1 Review

Do not assume the wave functions are normalized.

**R1** A 1eV free electron is incident on a nano-coating that is 10nm thick. While in the coating, the electron has potential energy 3eV. Compute the probability of the electron making it through the coating. One electron volt is $1\text{eV} = 1.602 \times 10^{-19}\text{J}$.

**R2** A helium atom has diameter $6.4 \times 10^{-11}$m. Compute the minimum speed of an electron in helium.

**R3** A particle with mass $m$ has wave function

$$\psi(x,0) = \begin{cases} 0 & \text{if } x < -a \\ 1 & \text{if } -a < x < 0 \\ 2 & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$$

Compute the average location of the particle and the uncertainly in the location of the particle.

**R4** A particle with mass $m$ has wave function

$$\psi(x,0) = \begin{cases} 0 & \text{if } x < -a \\ 1 & \text{if } -a < x < 0 \\ 2 & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$$

Compute the momentum distribution of the particle, $\phi^*(k)\phi(k)$.

**R5** A particle of mass $m$ is localized in an infinite square well of width $a$. The well extends from 0 to $+a$. The particle begins with the wave function $\psi(x,0) = \sin(2\pi x/a) + \sin(4\pi x/a)$. Compute the probability density of the wave function as a function of time. What are the allowed energy states and with what probability?

**R6** To a first approximation a metal cube can be treated as an infinite square well. Compute the ground state energy of an electron in a 10cm block of aluminum. (There is naturally a little more to this since electrons are fermions and the cube is three-dimensional.) Compare the velocity of the electron with the minimum velocity implied by uncertainty.

**R7** A particle is initially confined to the area $a/3$ to $2a/3$ with uniform probability. The confinement is removed releasing the particle into an infinite square well extending from $0 < x < a$. Compute the probability of finding the particle in the ground state.

**R8** A delta function barrier, $V(x) = \alpha\delta(x)$ is placed in the center of an infinite square well from $-L < x < L$. Find the bound states. This one may be too hard.

**R9** A particle has wave function $\psi = 1/\cosh(bx)$ where $b$ is a constant. Show all the expressions you would have calculate to verify the uncertainty relation for the particle. The expressions should be reduced as far as possible without performing an integration.

**R10** A particle of mass $m$ is incident on a delta function barrier at the origin, $V(x) = \alpha\delta(x)$, and a finite barrier of height $V_0$ from $x = b$ to $x = c$. Write the wave function and all boundary conditions the system must satisfy. The energy of the particle is less than $V_0$.

**R11** A particle is in the state

$$\psi(x,0) = \begin{cases} 0 & \text{if } x < -a \text{ or } x > a \\ A(a^4 - x^4) & \text{if } -a < x < a \end{cases}$$

What is the probability of finding the particle at $x > a/2$? What is the uncertainty in the particles momentum?

① For a finite barrier, the transmission probability is

$$\frac{1}{T} = 1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2\left(\frac{2a}{\hbar}\sqrt{2m(V_0 - E)}\right)$$

$$V_0 = 3E$$

$$\frac{1}{T} = 1 + \frac{9E^2}{4E(3E - E)} \sinh^2\left(\frac{2a}{\hbar}\sqrt{2m(3E - E)}\right)$$

$$= 1 + \frac{9}{8}\sinh^2\left(\frac{2a}{\hbar}\sqrt{4mE}\right)$$

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$$\frac{4a}{\hbar}\sqrt{mE} = \frac{4(10\times10^{-9}\,m)}{1.05\times10^{-34}\,Js}\sqrt{9.11\times10^{-31}\,kg \cdot 1.602\times10^{-19}\,J}$$

$$= 146$$

$$\frac{1}{T} = 1 + \frac{9}{8}\sinh^2(146)$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

(Thanks Rachel) The well extends from $-a$ to $a$.

If the well is 10 nm thick, $a = 5\,nm$.

$$\frac{4a}{\hbar}\sqrt{mE} = \frac{4(5\times10^{-9}m)}{1.05\times10^{-34}Js}\sqrt{(9.11\times10^{-31}kg)(1.602\times10^{-19}J)}$$

$$= 73$$

$$\frac{1}{T} = 1 + \frac{9}{8}\sinh^2(73) = 1.78\times10^{30}$$

$$T = 5.6\times10^{-31}$$

R2

Let $\Delta x = 6.4 \times 10^{-11}\,m$

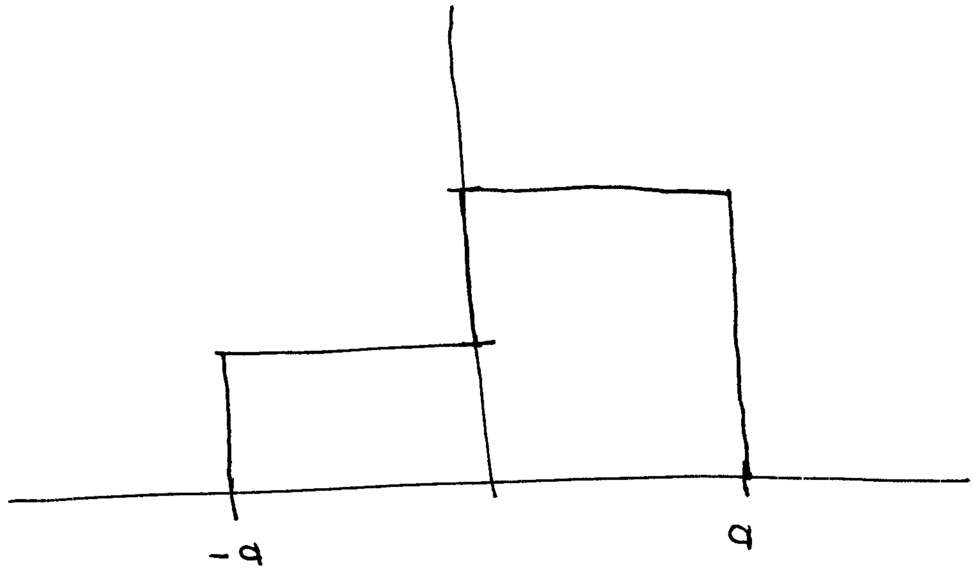
$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

$$\Delta p \geq \frac{\hbar}{2\Delta x}$$

$$\Delta v \geq \frac{\hbar}{2\Delta x\, m} = \frac{1.05 \times 10^{-34}\,Js}{2(6.4 \times 10^{-11}\,m)(\cancel{9 \times 10^{-3}}\,k \;\; 9.11 \times 10^{-31}\,kg)}$$

$$\geq 9 \times 10^{5}\,m/s$$

R3




<u>Normalize</u>

$$\psi^*\psi = A^2 \begin{cases} 0 & x<-a \\ 1 & -a<x<0 \\ 4 & 0<x<a \\ 0 & x>a \end{cases}$$

$$1 = \cancel{A^2}\int \psi^*\psi\, dx = A^2 \cdot (1\cdot a + 4\cdot a)$$

$$\frac{1}{5a} = A^2 \qquad A = \frac{1}{\sqrt{5a}}$$

$$\psi = \begin{cases} 0 & x<-a \\ A & -a<x<0 \\ 2A & 0<x<a \\ 0 & x>a \end{cases}$$

$$\langle x \rangle = \int \psi^* x \psi \, dx = A^2 \int_{-a}^{0} x \, dx + 4A^2 \int_{0}^{a} x \, dx$$

$$= A^2 \frac{x^2}{2} \Big|_{-a}^{0} + 4A^2 \frac{x^2}{2} \Big|_{0}^{a}$$

$$= -\frac{A^2 a^2}{2} + \frac{4A^2 a^2}{2}$$

$$= \frac{3}{2} A^2 a^2 = \frac{3}{10} a$$ which is at least reasonable

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$$\langle x^2 \rangle = \int \psi^* x^2 \psi \, dx = A^2 \int_{-a}^{0} x^2 \, dx + 4A^2 \int_{0}^{a} x^2 \, dx$$

$$= A^2 \frac{x^3}{3} \Big|_{-a}^{0} + 4A^2 \frac{x^3}{3} \Big|_{0}^{a}$$

$$= \frac{A^2 a^3}{3} + \frac{4A^2 a^3}{3} = \frac{5A^2 a^3}{3} = \frac{a^2}{3}$$

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$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{a^2}{3} - \left(\frac{3}{10} a\right)^2}$$

$$= a \sqrt{\frac{1}{9} - \frac{9}{100}} = \frac{\sqrt{19}}{30} a = 0.38 a$$

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{a^2}{3} - \left(\frac{3a}{10}\right)^2}$$

$$= a\sqrt{\frac{1}{3} - \frac{9}{100}} = 0.493\,a$$

(R4) Use same normalization as in R3, $A = \sqrt{\frac{1}{5a}}$

$$\psi(x,0) = \begin{cases} 0 & x < -a \\ A & -a < x < 0 \\ 2A & 0 < x < a \\ 0 & x > a \end{cases}$$

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int \psi(x,0)\, e^{-ikx}\, dx$$

$$= \frac{A}{\sqrt{2\pi}} \left[ \int_{-a}^{0} e^{-ikx}\, dx + 2\int_{0}^{a} e^{-ikx}\, dx \right]$$

$$= \frac{A}{\sqrt{2\pi}} \left[ \frac{e^{-ikx}}{-ik} \Bigg|_{-a}^{0} + 2\, \frac{e^{-ikx}}{-ik} \Bigg|_{0}^{a} \right]$$

$$= \frac{A}{-ik\sqrt{2\pi}} \left[ 1 - e^{ika} + 2e^{-ika} - 2 \right]$$

$$= \frac{iA}{k\sqrt{2\pi}} \left[ 2e^{-ika} - e^{ika} - 1 \right]$$

$$\phi^* \phi = \frac{-iA}{k\sqrt{2\pi}}\left(2e^{ika} - e^{-ika} - 1\right) \cdot$$

$$\frac{iA}{k\sqrt{2\pi}}\left(2e^{-ika} - e^{ika} - 1\right)$$

$$= \frac{A^2}{k^2 2\pi}\Big(4 - 2\left(e^{ika} + e^{-ika}\right) + 1 + 1$$

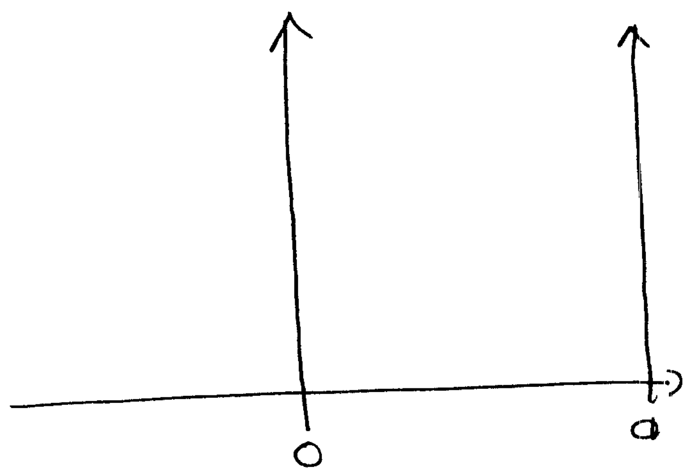
$$+ \left(e^{ika} + e^{-ika}\right)$$

$$-2\left(e^{2ika} + e^{-2ika}\right)\Big)$$

$$= \frac{A^2}{k^2 2\pi}\left(6 - 4\cos ka + 2\cos ka - 4\cos 2ka\right)$$

$$= \frac{A^2}{k^2 2\pi}\left(6 - 4\cos 2ka - 2\cos ka\right)$$

R5




$$\psi(x,0) = A\left(\sin\frac{2\pi x}{a} + \sin\frac{4\pi x}{a}\right)$$

$$= A'(\phi_2 + \phi_4) = A'\left(\sqrt{\frac{2}{a}}\sin\frac{2\pi x}{a} + \sqrt{\frac{2}{a}}\sin\frac{4\pi x}{a}\right)$$

where $\phi_i = \sqrt{\frac{2}{a}}\sin k_i x$

$$k_n = \frac{n\pi}{x}$$

are the stationary states of the infinite square well.

Normalize, do the integral and realize $A = \frac{1}{\sqrt{2}}$

$$\psi(x,0) = \frac{1}{\sqrt{2}}\phi_2 + \frac{1}{\sqrt{2}}\phi_4$$

The allowed energies are

$$E_2 = \frac{k_2^2 \hbar^2}{2m} = \left(\frac{\hbar^2}{2m}\right)\frac{2^2\pi^2}{a^2}$$

$$= 4\left(\frac{\pi^2\hbar^2}{2m}\right) = 4E_1$$

$$E_4 = 16E_1$$

These energies will be observed with equal probability

$$P_2 = P_4 = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$$

Let $\omega = \frac{E_1}{\hbar}$, then $\omega_2 = 4\omega$ and $\omega_4 = 16\omega$.

The time dependent wave function is

$$\psi(x,t) = \frac{1}{\sqrt{2}}\phi_2 e^{-4i\omega t} + \frac{1}{\sqrt{2}}\phi_4 e^{-16i\omega t}$$

$$\rho(x,t) = \psi^* \psi$$

$$= \frac{1}{2}\left( \phi_2^* e^{4i\omega t} + \phi_4^* e^{16i\omega t} \right) \cdot \left( \phi_2 e^{-4i\omega t} + \phi_4 e^{-16i\omega t} \right)$$

$$= \frac{1}{2}\left( \phi_2^2 + \phi_4^2 + \phi_2 \phi_4 e^{-12i\omega t} + \phi_2 \phi_4 e^{12i\omega t} \right)$$

since $\phi_i$ real.

$$\frac{e^{ix} + e^{-ix}}{2} = \cos x$$

$$\rho(x,t) = \frac{1}{2}\left( \phi_2^2 + \phi_4^2 + 2\phi_2\phi_4 \cos 12\omega t \right)$$

$$= \frac{1}{a}\left( \sin^2\left(\frac{2\pi x}{a}\right) + \sin^2\left(\frac{4\pi x}{a}\right) + 2\sin\frac{2\pi x}{a} \sin\frac{4\pi x}{a} \cdot \cos 12\omega t \right)$$

R6 Uncertainty $\Delta x \Delta p = m \Delta v \Delta x \geq \frac{\hbar}{2}$

$$\Delta v = \frac{\hbar}{2m\Delta x} = \frac{1.05 \times 10^{-34}\ J \cdot s}{2(9.11 \times 10^{-31} kg)(0.1m)}$$

$$= 6 \times 10^{-4}\ m/s$$

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Ground state $a = 10\ cm$

$$E_1 = \frac{\hbar^2 k_1^2}{2m} = \frac{\hbar^2 \pi^2}{2ma^2}$$

$$= \frac{(1.05 \times 10^{-34} J \cdot s)^2 \pi^2}{2(9.11 \times 10^{-31} kg)(0.1m)^2}$$

$$= 6 \times 10^{-36}\ J = \frac{1}{2} m v^2$$

$$v = \sqrt{\frac{2E}{m}} = 3.6 \times 10^{-3} m/s$$

a bit faster.

R7

$$\psi(x,0) = \begin{cases} 0 & x < a/3 \\ A & a/3 < x < 2a/3 \\ 0 & x > 2a/3 \end{cases}$$

<u>Normalize</u>

~~$1 = A^2 \int_{2a/3}$~~

$$1 = \int \psi^* \psi \, dx$$

$$= A^2 \int_{a/3}^{2a/3} dx$$

$$= A^2 \left( \frac{2a}{3} - a/3 \right)$$

$$A = \sqrt{\frac{3}{a}}$$

The ground state wave function is

$$\phi_1 = \sqrt{\frac{2}{a}} \sin \frac{\pi x}{a}$$

<u>Expand $\psi$</u>

$$\psi = \sum c_i \phi_i$$

## Use Fourier's Trick

$$c_1 = \int \phi_1^* \psi \, dx$$

$$= \sqrt{\frac{3}{a}}\sqrt{\frac{2}{a}} \int_{a/3}^{2a/3} \sin\frac{\pi x}{a}\, dx$$

$$= -\frac{1}{a}\sqrt{6}\left(\frac{a}{\pi}\right) \cos\frac{\pi x}{a}\Bigg|_{a/3}^{2a/3}$$

$$\cancel{= \pi\sqrt{6}}$$

$$c_1 = \frac{\sqrt{6}}{\pi}\left(\cos\frac{\pi}{3} - \cos\frac{2\pi}{3}\right)$$

$$= \frac{\sqrt{6}}{\pi}\left(\frac{1}{2} - \left(-\frac{1}{2}\right)\right)$$

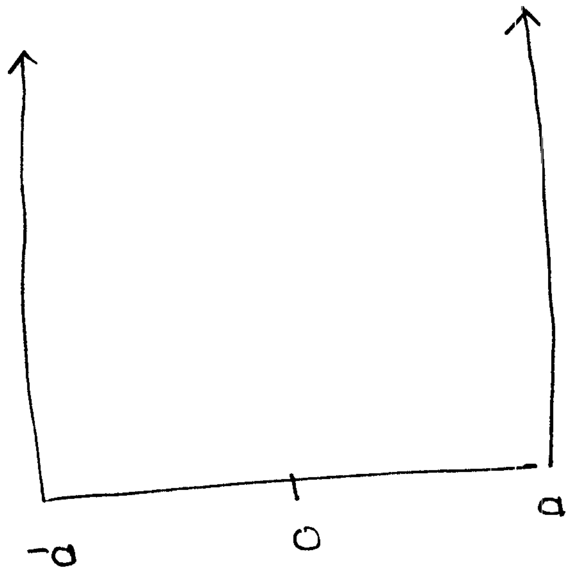
$$c_1 = \frac{\sqrt{6}}{\pi}$$

The probability of being in the ground state is

$$P_1 = c_1^* c_1 = \frac{6}{\pi^2} = 0.61$$

R8

Infinite Square Well $[-a, a]$




Now, even solutions ~~A sin kx~~ $A \cos kx$

and odd solutions, $A \sin kx$.

Both have to be zero at boundary,

$$A \sin ka = 0 \quad \Rightarrow \quad k = \frac{n\pi}{a} \qquad n = 1, 2, \cdots$$

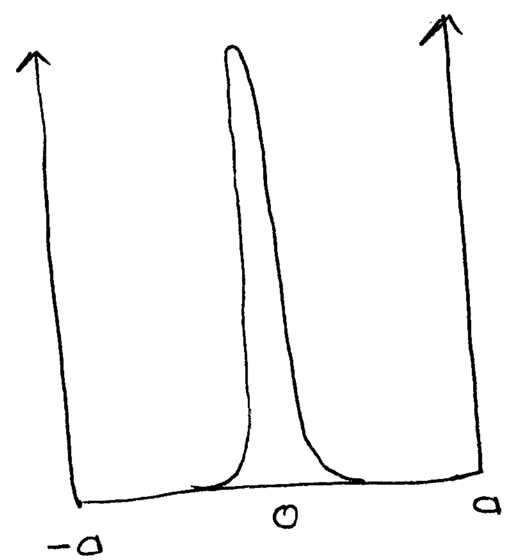
$$A \cos ka = 0 \qquad k = \cancel{\tfrac{1}{2}}\left(n + \tfrac{1}{2}\right)\frac{\pi}{a} \qquad n = 0, 1, 2 \cdots$$

<u>Normalization</u> $\sqrt{\frac{1}{a}}$

odd $\quad \phi = \sqrt{\frac{1}{a}} \sin \frac{n\pi}{a} x \qquad n = 1, 2, \cdots$

even $\quad \phi = \sqrt{\frac{1}{a}} \cos\left(n + \frac{1}{2}\right)\frac{\pi x}{a} \qquad n = 0, 1, \cdots$

Now add delta function



The odd solutions are not affected by the delta function because

$$\Delta \frac{d\phi}{dx} = \frac{2m\alpha}{\hbar^2}\phi(0) = 0$$

The even solutions will be changed. Start with the general solution.

$$\phi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x<0 \\ Ce^{ikx} + De^{-ikx} & x>0 \end{cases}$$

<u>Apply even condition</u> $\quad \phi(x) = \phi(-x)$

$$\Longrightarrow \quad A = D \qquad B = C$$

$$\phi(x) = \begin{cases} A\,e^{ikx} + Be^{-ikx} \\ Be^{ikx} + A\,e^{-ikx} \end{cases}$$

<u>Continuity</u> $(x=0)$ $\quad A+B = A+B$ ✓

$\phi(a) = 0$

$$Be^{ika} + Ae^{-ika} = 0$$

$$B = -Ae^{-2ika} \qquad (1)$$

Slope at $x=0$

$$\Delta \frac{d\phi}{dx} = \frac{2m\alpha}{\hbar^2}\phi(0)$$

$$ikB - ikA - (ikA - ikB) = \frac{2m\alpha}{\hbar^2}(A+B)$$

$$B - A = \frac{m\alpha}{ik\hbar^2}(A+B)$$

$$B\left(1 + \frac{im\alpha}{k\hbar^2}\right) = A\left(1 - \frac{im\alpha}{k\hbar^2}\right) \qquad (2)$$

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$$\cancel{(1)/(2) \quad 1 + \frac{im\alpha}{k\hbar^2} = -\left(1 - \frac{im\alpha}{k\hbar^2}\right)e^{-2ika}}$$

$$\cancel{(1+iB)e^{ika} = -(1-iB)e^{-ika}}$$

2/1

$$1 + \frac{i m \alpha}{k \hbar^2} = -\left(1 - \frac{i m \alpha}{k \hbar^2}\right) e^{i 2 k a}$$

$$B \equiv \frac{m\alpha}{k\hbar^2}$$

$$\frac{1+iB}{1-iB} = -e^{+2ika}$$

$$\left| \frac{1+iB}{1-iB} \right| = 1 \qquad \frac{1+iB}{1-iB} = e^{i2\theta}$$

$$\cancel{\tan\theta = \frac{\quad}{1+B^2}}$$

Define $z = 1+iB = \rho e^{i\theta}$

$$\frac{z}{z^*} = e^{i2\theta}$$

$$\tan\theta = B/1$$

$$e^{2i\theta} = -e^{+2ika} = e^{-2ika + i\pi}$$

$$2i\theta = +2ika + i\pi$$

$$\boxed{2\tan^{-1}(B) = 2ka + \pi}$$

Fixes other energies.
A little too hard.

R9 $\qquad \psi(x,0) = A\,\text{sech}(bx)$

1 <u>Normalize</u>

$$1 = A^2 \int_{-\infty}^{\infty} \text{sech}^2(bx)\, dx$$

2 $$\langle x \rangle = \cancel{A}\int_{-\infty}^{\infty} x\, \psi^* \psi\, dx = A^2 \int_{-\infty}^{\infty} x\, \text{sech}^2(bx)\, dx$$

3 $$\langle x^2 \rangle = A^2 \int_{-\infty}^{\infty} x^2\, \text{sech}^2(bx)\, dx$$

4 $$\langle p \rangle = \int \psi^* \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right) \psi\, dx$$

$$\frac{\partial}{\partial x} \text{sech}(bx) = -b\, \text{sech}(bx) \tanh(bx)$$

$$\langle p \rangle = i b \hbar A^2 \int_{-\infty}^{\infty} \text{sech}^2(bx) \tanh(bx)\, dx$$

5 $\langle p^2 \rangle = \int \psi^* \left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)^2 \psi \, dx$

$$\frac{\partial^2}{\partial x} \text{sech}(bx) = -b\frac{\partial}{\partial x} \text{sech}(bx)\tanh(bx)$$

$$= -b\tanh(bx)\left[-b\,\text{sech}(bx)\tanh(bx)\right]$$

$$-b\,\text{sech}(bx)\left[b\,\text{sech}^2(bx)\right]$$

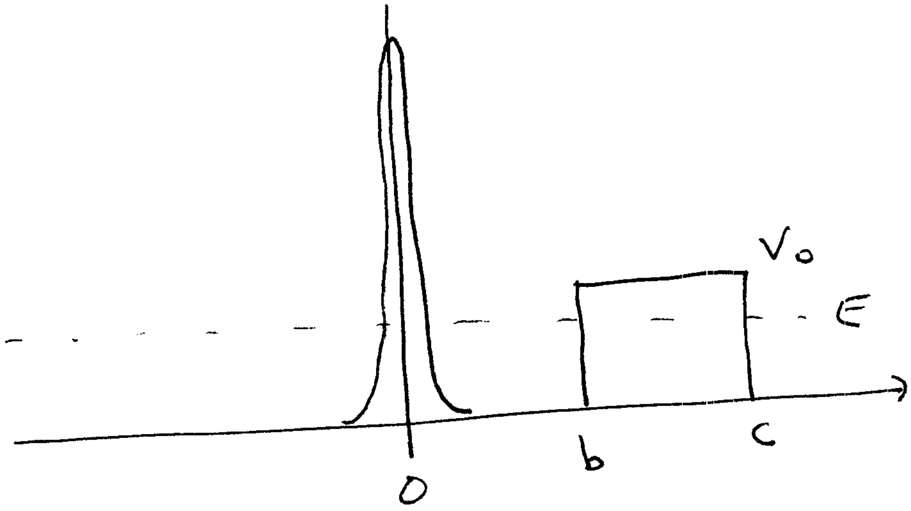
$$= b^2\left(\text{sech}(bx)\tanh^2(bx) - \text{sech}^3(x)\right)$$

$$\langle p^2 \rangle = -\hbar^2 b^2 A^2 \int_{-\infty}^{\infty}\left[\text{sech}^2(bx)\tanh^2(bx) - \text{sech}^4(bx)\right]dx$$

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \qquad \sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$$

Uncertainty $\sigma_x \sigma_p \geq \frac{\hbar}{2}$

R10




$$\phi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & x<0 \\ Ce^{ikx} + De^{-ikx} & 0<x<b \\ Ee^{Kx} + Fe^{-Kx} & b<x<c \\ Ge^{ikx} & x>c \end{cases}$$

$$k = \sqrt{\frac{2mE}{\hbar^2}} \qquad K = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

Continuity

$\underline{x=0}$ $\quad A + B = C + D$

$\underline{x=b}$ $\quad Ce^{ikb} + De^{-ikb} = Ee^{Kb} + Fe^{-Kb}$

$\underline{x=c}$ $\quad Ee^{Kc} + Fe^{-Kc} = Ge^{ikc}$

## Slope Continuous

$\underline{x=b}$

$$ike^{ikb}C - ikDe^{-ikb} = \kappa E e^{\kappa b} - \kappa F e^{-\kappa b}$$

$\underline{x=c}$

$$\kappa E e^{\kappa c} - \kappa F e^{-\kappa c} = ikGe^{ikc}$$

## Delta Function

$$\left.\frac{d\phi}{dx}\right|_{+\varepsilon} - \left.\frac{d\phi}{dx}\right|_{-\varepsilon} = \frac{2m\alpha}{\hbar^2}\phi(0)$$

$$(ikA - ikB) - (\kappa C - \kappa D) = \frac{2m\alpha}{\hbar^2}(A+B)$$

R11 Normalize

$$1 = A^2 \int (a^4 - x^4)^2 \, dx$$

$$= A^2 \left( \frac{64}{45} a^9 \right)$$

$$A = \sqrt{\frac{45}{64 a^9}}$$

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$$P(x > a/2) = \int_{a/2}^{a} \psi^* \psi \, dx$$

$$= A^2 \int_{a/2}^{a} (a^4 - x^4)^2 \, dx$$

$$= \frac{5147}{32768}$$

$$\langle p \rangle = \int \psi^* \left( \frac{\hbar}{i} \frac{\partial \psi}{\partial x} \right) dx$$

$$= \frac{\hbar}{i} A^2 \int (a^4 - x^4) \frac{d}{dx} (a^4 - x^4) \, dx$$

$$= 0$$

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$$\langle p^2 \rangle = \int \psi^* \left( \frac{\hbar}{i} \frac{\partial}{\partial x} \right)^2 \psi \, dx$$

$$= -\hbar^2 A^2 \int (a^4 - x^4) \frac{d^2}{dx^2} (a^4 - x^4) \, dx$$

$$= \frac{45}{14} \frac{\hbar^2}{a^2}$$

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$$\sigma_p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \frac{\hbar}{a} \sqrt{\frac{45}{14}}$$

> $int\left(\left(a^4 - x^4\right)^2,\ x = -a..a\right);$

$$\frac{64}{45}\ a^9 \tag{1}$$

> $A := \left(\frac{45}{\left(64 \cdot a^9\right)}\right)^{\left(\frac{1}{2}\right)};$

$$A := \frac{1}{64}\sqrt{45}\ \sqrt{64}\ \sqrt{\frac{1}{a^9}} \tag{2}$$

> $int\left(A^2 \cdot \left(a^4 - x^4\right)^2,\ x = \frac{a}{2}..a\right);$

$$\frac{5147}{32768} \tag{3}$$

> $diff\left(A^2 \cdot \left(a^4 - x^4\right)^2,\ x\right);$

$$-\frac{45}{8}\ \frac{\left(a^4 - x^4\right)x^3}{a^9} \tag{4}$$

> $int\left(A^2 \cdot \left(a^4 - x^4\right) \cdot diff\left(\left(a^4 - x^4\right),\ x\right),\ x = -a..a\right);$

$$0 \tag{5}$$

> $int\left(A^2 \cdot \left(a^4 - x^4\right) \cdot diff\left(diff\left(\left(a^4 - x^4\right), x\right), x\right),\ x = -a..a\right);$

$$-\frac{45}{14}\ \frac{1}{a^2} \tag{6}$$

>